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N=2 Quantum String Scattering

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Abstract

We calculate the genus-one three- and four-point amplitudes in the 2+2 dimensional closed $N=(2,2)$ worldsheet supersymmetric string within the RNS formulation. Vertex operators are redefined with the incorporation of spinor helicity techniques, and the quantum scattering is shown to be manifestly gauge and Lorentz invariant after normalizing the string states. The continuous spin structure summation over the monodromies of the worldsheet fermions is carried out explicitly, and the field-theory limit is extracted. The amplitude in this limit is shown to be the maximally helicity violating amplitude in pure gravity evaluated in a two-dimensional setting, which vanishes, unlike the four-dimensional result. The vanishing of the genus-one $N=2$ closed string amplitude is related to the absence of one-loop divergences in dimensionally regulated IIB supergravity. Comparisons and contrasts between self-dual field theory and the $N=2$ string theory are made at the quantum level; they have different S-matrices. Finally, we point to further relations with self-dual field theory and two-dimensional models.

1 Introduction

The field equations

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} \qquad F_{\mu\nu} = \tilde{F}_{\mu\nu} \qquad (1.1)$$

pertain to many aspects of physics and mathematical physics: self-dual field theory, string theory, instantons and monopoles, and the classification of four-manifolds.

The $N=2$ worldsheet supersymmetric string is unique among string theories as its critical dimension is four [1, 2, 3, 4, 5]. Its full spectrum and its exact (in α') classical field equations have been identified to be merely those of self-dual gravity and self-dual Yang-Mills theory [6, 7] (for a review up to 1992, see [8]). Several quantum field theory formulations of the latter theories point to a non-vanishing S-matrix at the quantum level [9], a fact which is, however, in contrast to the claims of zero quantum S-matrix for the $N=2$ string [10, 11, 12]. Apparently, we witness a quantum discrepancy between two theories which are classically equivalent. In this work we address this question by calculating the $N=(2,2)$ closed-string genus-one amplitudes in the RNS formulation and identifying the target spacetime theory which gives rise to these amplitudes.

Self-duality in $d = 2+2$ dimensions (or in $d = 4+0$) is implemented in the field equations of gravity and Yang-Mills by (1.1), of which the only known Lorentz covariant Lagrangian formulation employs Lagrange multipliers and is given by ¹

$$\mathcal{L} = \text{Tr } G^{\alpha\beta} F_{\alpha\beta} \qquad \mathcal{L} = \text{Tr } \rho^{\alpha\beta} R_{\alpha\beta} \quad , \qquad (1.2)$$

where F and R are the self-dual projections of the field-strength and Riemann tensor for a gauge and spin connection vector, respectively. [13, 9]. The Lagrangian (1.2) involves two fields, related to the two polarization states of a gauge field, yet only one appears as an asymptotic state.

Alternatively, fixing a light-cone gauge in (1.1) allowed Leznov [14] and Plebanski [15] to reduce the self-duality equations to a pair of second-order equations

$$-\square\phi + \frac{g}{2}[\partial_+ \dot{\alpha}\phi, \partial_{+\dot{\alpha}}\phi] = 0 \qquad -\square\psi + \frac{\kappa}{2}\partial_+ \dot{\alpha}\partial_+ \dot{\beta}\psi \partial_{+\dot{\alpha}}\partial_{+\dot{\beta}}\psi = 0 \qquad (1.3)$$

for scalar prepotentials ϕ (to F) and ψ (to R) which extremize the respective Lorentz non-covariant gauge-fixed actions belonging to

$$\mathcal{L} = \text{Tr } \phi \left(-\frac{1}{2}\square\phi + \frac{g}{6}[\partial_+ \dot{\alpha}\phi, \partial_{+\dot{\alpha}}\phi] \right) \qquad (1.4)$$

and

$$\mathcal{L} = \psi \left(-\frac{1}{2}\square\psi + \frac{\kappa}{6}\partial_+ \dot{\alpha}\partial_+ \dot{\beta}\psi \partial_{+\dot{\alpha}}\partial_{+\dot{\beta}}\psi \right) . \qquad (1.5)$$

¹ Sub- and superscripts $\alpha \in \{+, -\}$ and $\dot{\alpha} \in \{\dot{+}, \dot{-}\}$ are spinor indices of $SL(2, \mathbb{R})$ (or $SU(2)$).

Further Lorentz non-covariant formulations of self-dual quantum field theories can be found by solving the gauge constraints in (1.1) differently [16, 17]. As these one-field actions share a coupling constant of positive length dimension they are all power-counting non-renormalizable.

The Lorentz-covariant two-field actions [9] are much better behaved in this respect. In light-cone gauge, their Lagrangians are

$$\mathcal{L} = \text{Tr } \tilde{\phi} \left(-\square \phi + \frac{g}{2} [\partial_+ \dot{\phi}, \partial_{+\dot{\alpha}} \phi] \right) \quad (1.6)$$

and

$$\mathcal{L} = \tilde{\psi} \left(-\square \psi + \frac{\kappa}{2} \partial_+ \dot{\psi} \partial_{+\dot{\beta}} \psi \partial_{+\dot{\alpha}} \partial_{+\dot{\beta}} \psi \right) \quad (1.7)$$

which allows no scattering beyond one-loop, because the multiplier fields go with $1/\hbar$. The one- and two-field theories both generate the maximally helicity violating (MHV) scattering at one-loop and the vanishing next-to-MHV amplitudes at tree-level [9], and the latter theories are one-loop exact perturbatively.

To compare with, the $N=2$ superstring has been shown (modulo contact term ambiguities) to possess trivial scattering in its critical dimension [11]. This indicates the presence of an anomaly in the string, or a target-space interpretation different from self-dual gravity or gauge theory. A possible anomaly interpretation behind the $d = 3+1$ MHV amplitudes in gauge theory was initially pointed out in [18] in the context of the conserved symmetries of the field equations.

Until now, the $N=2$ string quantum amplitude has never been computed in the traditional RNS formalism (but functional methods for the quantization at higher genera have been developed [19]). However, by embedding the $N=2$ string in an $N=4$ topological string it was demonstrated that, up to contact terms, these amplitudes vanish to all loops [10, 11]. Linearized symmetry arguments have also formally shown this in the RNS formulation in [12]. In order to compare with the field-theory results and to find the root of this discrepancy, an explicit traditional computation at genus one is worthwhile. In the present work we perform this calculation. We find that the $N=2$ string loop dynamics appears to be reduced to two dimensions. Based on earlier one-loop computations of the partition function [20] and the three-point function [21], Marcus [8] already identified the technical origin of this dimensional mismatch. Here, we confirm his observation and extend it to the full quantum dynamics by evaluating the one-loop four-point scattering. As the $N=2$ string has a critical dimension of four, with four (real) target spacetime coordinates, this calculation indicates that it represents a ghost system in the MHV sector of gauge theory. In order to make the above explicit we render the scattering manifestly Lorentz invariant by normalizing the vertex operators and incorporating the gauge invariance through the use of spinor helicity techniques.

A further unexpected relation arises between the one-loop MHV amplitude in pure gravity regulated to two dimensions and the next-to-MHV amplitude in IIB

supergravity evaluated in ten dimensions.² The absence of one-loop divergences in the massless sector of IIB supergravity in ten dimensions within dimensional reduction explains the vanishing of the two-dimensional MHV result and thereby the triviality of the field theory limit of the $N=2$ string scattering. Alternatively, a relation is found between two string theories: the IIB superstring in ten dimensions and the $N=2$ closed string in four dimensions. Such a relation may originate in an integrable structure in the ultra-violet regime for the massless modes of the string (a supergravity analog of the Regge kinematical limit of Yang-Mills theory). If this connection extends to multi-loops, the vanishing theorems of the $N=2$ string at higher genera deserve further study.

The outline of this work is as follows. In section 2 we review and discuss the properties of the gauge theory MHV amplitudes in different dimensions. Section 3 implements gauge invariance directly into the $N=2$ string scattering through the incorporation of spinor helicity techniques and a normalization of the vertex operators. We also analyze contact term subtleties in the scattering. In section 4 the three-point genus-one closed-string amplitude is obtained and compared with the field theoretic one. In section 5 we finally compute a modular-integral expression for the genus-one closed-string four-point amplitude, carefully taking into account the superconformal ghost structure. From this result, we extract the field theory limit by taking $\alpha' \rightarrow 0$ before summing over spin structures (which then trivializes). The answer is zero, and the comparison with the MHV amplitudes is made. In section 6 we explicitly perform the spin structure summation on the full modular integrand before taking the field theory limit, with identical (vanishing) result. A discussion and an Appendix on Jacobi theta functions conclude the paper.

2 Review of MHV Gauge Theory Amplitudes

Recent developments in techniques in gauge theory calculations³ have made possible the calculation of closed analytic forms of several infinite sequences of one-loop gauge theory amplitudes. The maximally helicity violating (MHV) amplitudes are described by scattering of gauge fields of identical helicity, either in Yang-Mills theory or in gravity. One of the features of these amplitudes is that in a supersymmetric theory they are identically zero to infinite loop order; this implies that at tree-level the amplitudes are identically zero. The amplitudes closest to MHV are simpler to calculate, and the self-dual description has lead to reformulations and improved diagrammatic techniques in calculating gauge theory amplitudes [25]

² This dimension-shifting relation involving a change in the number of supersymmetries was initially found in [22].

³ For a review at tree-level see [23] and at loop-level [24].

as well as second-order formulations for incorporating fermions ⁴ [26, 27]. In this section we briefly review these maximally helicity violating amplitudes and describe their relations to both self-dual field theory and string theory. The continuation of the four-point MHV gravity amplitude and its conjectured form to n -point order to arbitrary dimensions is directly related to the zero-slope limit of the $N=2$ closed string.

At one-loop in four dimensions, the leading-in-color partial amplitude for the scattering of n gluons of identical out-going helicity in Yang-Mills theory is [28, 29]

$$A_{n;1}^{[1]}(k_i) = -\frac{i}{48\pi^2} \sum_{1 \leq i < j < k < l \leq n} \frac{\langle ij \rangle [jk] \langle kl \rangle [li]}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \quad (2.1)$$

where the superscript $[J]$ represents the spin of the internal state (gluon, complex scalar or Weyl fermion give the same result up to a minus sign for half-integral spin). The amplitude is written in color-ordered form [30]; the leading-in-color group theory structure,

$$N^2 \text{Tr } T^{a_1} T^{a_2} \cdots T^{a_n} , \quad (2.2)$$

has been extracted from the kinematics in accord with Chan-Paton assignments in open string theory and gauge theory. In (2.1) we have decomposed each lightlike momentum vector k_i into two momentum Weyl spinors and defined two different inner products,

$$k_i^{\alpha\dot{\alpha}} = k_i^\alpha k_i^{\dot{\alpha}} \quad \text{and} \quad \langle ij \rangle = k_i^\alpha k_{j,\alpha} \quad [ij] = k_i^{\dot{\alpha}} k_{j,\dot{\alpha}} . \quad (2.3)$$

In 2+2 dimensions, $\langle ij \rangle$ is not the complex conjugate of $[ij]$; the Lorentz group is $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})'$ as opposed to $SL(2, \mathbb{C})$ in 3+1 dimensions (and there are poles in (2.1) in the self-dual plane parameterized by the $SL(2, \mathbb{R})$ half of the Lorentz group).

The analogous result for the all-plus gravitational amplitude [31, 32],

$$A_4^{[2]}(k_i) = -i \left(\frac{\kappa}{2} \right)^4 \frac{1}{120 (4\pi)^2} \left(\frac{s_{12} s_{23}}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \right)^2 (s_{12}^2 + s_{23}^2 + s_{13}^2) , \quad (2.4)$$

and its n -point form [33], together with the three-point vertex, describe the scattering of all-plus helicity gravitons to one-loop order. The amplitude in (2.1) and its gravitational analog have a number of features in common with $N=2$ string scattering. They are channel-dual in the sense that exchange of any two legs gives the same form. Furthermore, they have only two-particle poles (in one $SL(2, \mathbb{R})$ factor of the Lorentz group), which signals integrable characteristics related to the infinite number of symmetries in the self-dual field equations.

⁴ In a self-dual non-abelian background, fermions may be bosonized, and fields become spin independent.

The n -point gauge theory amplitude in (2.1) has been found by constraining the functional form based on analyticity [28] as well as through a direct calculation with a fermion in the loop [29]. The amplitude in (2.1) also arises in a one-loop S-matrix element for self-dual Yang-Mills theory. This happens for the Lorentz-covariant two-field theory (one-loop exact) [9] as well as, to a factor of two, for the one-field (Leznov) formulation [9, 34], although the latter is not Lorentz covariant (or one-loop exact). The same story occurs in gravity [33] where (2.4) and its generalizations describe quantum self-dual gravity at one-loop [9]. One might expect to find similar non-vanishing scattering amplitudes at the one-loop level in the $N=2$ string, as both the string and the self-dual field theory share the same classical field equations. However, this expectation is not borne out by our calculation below.

The d dimensional generalization of the Yang-Mills result in (2.1) has been found in [22] up to six-point (together with a conjectured form at $n \geq 7$ point), and we list here the form of these amplitudes. At four-point one has

$$A_{4;1}^{[1]}(k_i) = \frac{-2i}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{(4-d)(2-d)}{4(4\pi)^2} s_{12}s_{23} I_4^{4+d}(s, t), \quad (2.5)$$

where the box diagram I_4^{4+d} is the integral function

$$I_4^p(s, t) = \int \frac{d^p \ell}{(2\pi)^p} \frac{1}{\ell^2(\ell - k_1)^2(\ell - k_1 - k_2)^2(\ell + k_4)^2}, \quad (2.6)$$

continued from p to $d+4$ dimensions but with the external vectors in d dimensions. The generalization of the series in (2.1) arises by keeping the external kinematics and polarizations in four dimensions and analytically continuing the scalar integral functions. In a Schwinger proper-time formulation of the integrals this amounts to inserting additional factors of τ_2 in the integral over the proper time. In *four* dimensions, the 8-dimensional box diagram in (2.5) relevant to the amplitude is UV divergent, but the result is finite because the $d-4$ prefactor extracts the residue. In *two* dimensions the 6-dimensional box diagram with external massless kinematics is both IR and UV finite, but the prefactor forces the result in (2.5) to be identically zero. The MHV amplitude thus vanishes upon continuation to $d=2$, without recourse to spacetime supersymmetry.

The five- and six-point amplitudes and their dimensional form have the same properties as the expression (2.5), as does the conjectured n -point form at one-loop. The five-point amplitude,

$$\begin{aligned} A_{5;1}^{[1]}(k_i) = & \frac{-i}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \frac{(4-d)(2-d)}{4(4\pi)^{d/2}} \left[s_{23}s_{34}I_4^{d+4} + s_{34}s_{45}I_4^{d+4} \right. \\ & \left. + s_{45}s_{51}I_4^{d+4} + s_{51}s_{12}I_4^{d+4} + s_{12}s_{23}I_4^{d+4} + 4id\epsilon_{\mu\nu\rho\sigma}k_1^\mu k_2^\nu k_3^\rho k_4^\sigma I_5^{d+6} \right], \end{aligned} \quad (2.7)$$

is zero when continued to two dimensions because the six-dimensional box and eight-dimensional pentagon in (2.7) are finite and the pre-factor vanishes in $d=2$. The

gauge theory result at six-point is similar and is described in eqs. (16) and (17) of reference [22].

In (spacetime) *supersymmetric* gauge or gravitational theory, the MHV one-loop amplitudes vanish because of a cancellation between the contributions stemming from different spin states running inside the loop [35]. Concretely,

$$A^{[1]} = A^{[0]} = -A^{[\frac{1}{2}]} = A^{[2]}, \quad (2.8)$$

for a gauge boson, complex scalar, Weyl fermion, or graviton, so that amplitudes need to be computed for only one conveniently chosen spin value.

The all- n conjectured form of the MHV Yang-Mills amplitude relates to a $d+4$ $\mathcal{N}=16$ supersymmetric non-MHV amplitude as follows,

$$A_{n;1}^{[0]}(k_i) \Big|_d = \frac{(4-d)(2-d)}{2} (4\pi)^2 \frac{1}{\langle 12 \rangle^4} A_{n;1}^{\mathcal{N}=16}(k_1^-, k_2^-, k_3^+, \dots, k_n^+) \Big|_{d+4}, \quad (2.9)$$

where for definiteness we denote it for an internal complex scalar, with $[J=0]$. The factor of $\langle 12 \rangle^4$ gives the left-hand side the appropriate spinor weight to describe the negative-helicity gluons on legs one and two. Curiously, the prefactor in (2.9) is negative for $2 \leq d \leq 4$. Again, the finiteness of the amplitude on the right-hand side of (2.9) in $d+4=6$ translates into the vanishing of the MHV amplitude in $d=2$. For $d+4=8$ the UV singularity of $A_n^{\mathcal{N}=16}$ reproduces (2.1).

The explicit result in d dimensions for the four-point one-loop maximally helicity violating Einstein-Hilbert gravitational amplitude is

$$A_4^{[2]}(k_i) \Big|_d = \frac{(4-d)(2-d)d(2+d)}{8} (4\pi)^4 \frac{1}{\langle 12 \rangle^8} A_4^{\mathcal{N}=32}(1^{--}, 2^{--}, 3^{++}, 4^{++}) \Big|_{d+8} \quad (2.10)$$

where the relation is between an MHV amplitude in d dimensions to a non-MHV amplitude in $d+8$ dimensions and in the $\mathcal{N}=32$ (maximally) supersymmetric theory. Similar to the Yang-Mills case, the additional $\langle 12 \rangle^4$ gives the MHV amplitude the proper helicity weight (the graviton has twice the spin) and dimensions.

For $d=2$ the amplitude on the right-hand side in (2.10) is to be evaluated in ten dimensions. In this case no counterterms occur in the amplitude calculation in dimensional regularization since the divergences at four-point are proportional to

$$\left(\frac{1}{d-10} \right) (s+t) + \left(\frac{1}{d-10} \right) (t+u) + \left(\frac{1}{d-10} \right) (u+s) \quad (2.11)$$

which is zero on-shell, forcing the MHV result in (2.10) in $d=2$ to vanish. Parallel to the relation in (2.9) and generalizing (2.10), the conjectured d -dimensional gravitational MHV amplitude at arbitrary n -point coincides with the $\mathcal{N}=32$, $d+8$ next-to-MHV amplitude. The absence of a counterterm at n -point in the dimensionally regularized/reduced form of IIB supergravity in ten dimensions means that, due

to the prefactor in the n -point generalization of (2.10), the MHV result for graviton scattering in two dimensions is zero at arbitrary n -point order at one-loop.

Two-dimensional gravity and Yang-Mills theory are topological and have no dynamical degrees of freedom. The scattering in these theories is trivial in topologically trivial spacetime, which explains the vanishing of the amplitudes not only at one-loop but also to infinite loop order. A possible relation between the reduced form of the scattering in $d=2$ and that in $d=10$ implies further non-trivial structure in the ultra-violet of IIB supergravity.

In the following we shall relate the above $d=2$ result in the gravitational case to the scattering obtained in the RNS formulation of the closed $N=2$ superstring in the zero-slope limit. Given the holomorphic/anti-holomorphic factorization of the string integrand, this relation might persist to the open string as well.

3 N=2 String Vertex Operators

In this section we review the relevant facts of the closed $N=2$ string and its tree-level scattering amplitudes. We pay particular attention to the its vertex operators, for two reasons: First, the representation of the vertex operators affects possible contact interactions and their contributions to scattering amplitudes. Second, the normalization of the vertex operators translates to the choice of external leg factors which are crucial to achieve a manifestly gauge-invariant representation of the amplitudes via spinor helicity techniques. For a brief review, the reader may consult [36] and references therein.

3.1 Generalities

From the worldsheet point of view, critical closed $N=2$ strings in flat Kleinian space $\mathbb{R}^{2,2}$ are a theory of $N=(2,2)$ supergravity on a 1+1 dimensional (pseudo) Riemann surface, coupled to two chiral $N=(2,2)$ massless matter multiplets X^a , $a = 1, 2$. The latter's components are complex scalars x (the four string coordinates), $SO(1,1)$ Dirac spinors ψ (their four NSR partners) and complex auxiliaries F ,

$$X^a = x^a + \theta^- \psi^{+a} + \theta^+ \psi^{-\bar{a}} + \theta^+ \theta^- F^a \quad (3.1)$$

with arguments $y \equiv z + \theta^+ \theta^-$. Complex conjugation reads

$$z^* = \bar{z} \quad (\theta^+)^* = \bar{\theta}^- \quad (x^a)^* = \bar{x}^{\bar{a}} \quad (\psi^{+a})^* = \bar{\psi}^{-\bar{a}} \quad (3.2)$$

while chiral conjugation exchanges right- and left-movers via

$$z \rightarrow \bar{z} \quad \theta^\pm \rightarrow \bar{\theta}^\pm \quad x^a \rightarrow \bar{x}^{\bar{a}} \quad \psi^{+a} \rightarrow \bar{\psi}^{+\bar{a}} . \quad (3.3)$$

The extended worldsheet supersymmetry has induced a spacetime complex structure which reduces the global Lorentz symmetry,

$$\text{Spin}(2,2) = SU(1,1) \times SU(1,1)' \longrightarrow U(1) \times SU(1,1)' \simeq U(1,1) . \quad (3.4)$$

In superconformal gauge, however, manifest $SO(2, 2)$ symmetry is restored in the worldsheet action, which is given by

$$S = \int d^2z d^2\theta d^2\bar{\theta} K(X, \bar{X}) = \int d^2z \eta_{a\bar{a}} [\partial x^a \bar{\partial} x^{\bar{a}} + \psi^{+a} \bar{\partial} \psi^{-\bar{a}} + \bar{\psi}^{+a} \partial \bar{\psi}^{-\bar{a}}] \quad (3.5)$$

where $\eta_{a\bar{a}} = \text{diag}(+-)$ is the flat metric in $\mathbb{C}^{1,1}$, and the auxiliary fields have been integrated out.

Although the above notation makes transparent the local R symmetry properties of the fields (for instance, x is neutral while ψ^\pm is not), it is not convenient for our computations. The interrelation (3.2) with complex conjugation allows us to change it,

$$x^a \rightarrow x^{+a} \quad x^{\bar{a}} \rightarrow x^{-a} \quad \psi^{+a} \rightarrow \psi^{+a} \quad \psi^{-\bar{a}} \rightarrow \psi^{-a}, \quad (3.6)$$

so that the $SO(2, 2)$ invariant scalar product reads

$$k \cdot x = \frac{1}{2} (k^+ \cdot x^- + k^- \cdot x^+) = \frac{1}{2} (k^{+1} x^{-1} - k^{+2} x^{-2} + k^{-1} x^{+1} - k^{-2} x^{+2}) \quad (3.7)$$

where the dot is also used to denote the $SU(1, 1)'$ invariant scalar product. There exist three *antisymmetric* $SU(1, 1)'$ invariant products,

$$\begin{aligned} k^+ \wedge x^+ &= \epsilon_{ab} k^{+a} x^{+b} = k^{+1} x^{+2} - k^{+2} x^{+1} \\ k^+ \wedge x^- &= \frac{1}{2} (k^+ \cdot x^- - k^- \cdot x^+) = \frac{1}{2} (k^{+1} x^{-1} - k^{+2} x^{-2} - k^{-1} x^{+1} + k^{-2} x^{+2}) \\ k^- \wedge x^- &= \epsilon_{ab} k^{-a} x^{-b} = k^{-1} x^{-2} - k^{-2} x^{-1} \end{aligned} \quad (3.8)$$

which feature prominently in the following.

The $N=(2, 2)$ supergravity multiplet defines a gravitini and a Maxwell bundle over the worldsheet Riemann surface. The topology of the total space is labeled by the Euler number χ of the punctured Riemann surface and the first Chern number (instanton number) M of the Maxwell bundle. It is notationally convenient to replace the Euler number by the “spin”

$$J := -2\chi = 2n - 4 + 4(\#\text{handles}) \in 2\mathbb{Z} \quad . \quad (3.9)$$

The action (3.5) is to be considered for string worldsheets of a given topology.⁵ The first-quantized string path integral for the n -point function A_n includes a sum over worldsheet topologies (J, M) , weighted with appropriate powers in the string couplings $(\kappa, e^{i\theta})$:

$$A_n(\kappa, \theta) = \sum_{J=2n-4}^{\infty} \kappa^{J/2} A_n^J(\theta) = \sum_{J=2n-4}^{\infty} \sum_{M=-J}^{+J} \kappa^{J/2} e^{iM\theta} A_n^{J,M} \quad (3.10)$$

⁵ Of course, the Lagrangian in (3.5) is in general not correct globally.

where the instanton sum has a finite range because bundles with $|M| > J$ do not contribute. The presence of Maxwell instantons breaks the explicit $U(1)$ factor in (3.4) but the $SU(1,1)$ factor (and thus the whole $\text{Spin}(2,2)$) is fully restored if we let $\kappa^{1/4}(e^{i\theta/2}, e^{-i\theta/2})$ transform as an $SU(1,1)$ spinor. As a consequence, the string couplings depend on the $SO(2,2)$ Lorentz frame, and we may choose a convenient one for calculations. We call the choice $\theta=0$ a ‘Leznov frame’ and name an averaging over θ a ‘Yang frame’. The partial amplitudes $A_n^{J,M}$ are integrals over the metric, gravitini, and Maxwell moduli spaces. The integrands may be obtained as correlation functions of vertex operators in the $N=(2,2)$ superconformal field theory on the worldsheet surface of fixed shape (moduli) and topology.

The vertex operators produce from the (first-quantized) vacuum state the asymptotic string states in the scattering amplitude under consideration. They correspond to the physical states of the $N=2$ closed string and carry their quantum numbers. Being representatives of the (semi-chiral) BRST cohomology, they are unique only up to BRST-trivial terms and normalization. The physical subspace of the $N=2$ string Fock space in a covariant quantization scheme turns out to be surprisingly small [37]: Only the ground state $|k\rangle$ remains, a scalar on the massless level, i.e. for center-of-mass momentum $k^{\pm a}$ with $k \cdot k = 0$. The dynamics of this string “excitation” is described by a massless scalar field,

$$\Phi(x) = \int d^4k e^{-ik \cdot x} \tilde{\Phi}(k) , \quad (3.11)$$

whose self-interactions are determined on-shell from the (amputated) string scattering amplitudes at tree-level,

$$\langle \tilde{\Phi}(k_1) \dots \tilde{\Phi}(k_n) \rangle_{\text{tree}, \theta}^{\text{amp}} =: A_n^{2n-4}(k_1 \dots k_n; \theta) =: \delta_{k_1 + \dots + k_n} \tilde{A}_n^{2n-4}(k_1 \dots k_n; \theta) . \quad (3.12)$$

Interestingly, it has been shown that all tree-level n -point functions vanish on-shell, except for the three-point amplitude [38],

$$\tilde{A}_3^2(k_1, k_2, k_3; \theta) = -\frac{1}{4} \left[e^{i\theta} k_1^+ \wedge k_2^+ - 2 k_1^+ \wedge k_2^- - e^{-i\theta} k_1^- \wedge k_2^- \right]^2 \quad (3.13)$$

with $k_i \cdot k_j = 0$ due to $\sum_n k_n = 0$. Note that \tilde{A}_3^2 is totally symmetric in all momenta. Expanding the square, one reads off $\tilde{A}_3^{2,M}$ for $M=-2, \dots, +2$. However, using the on-shell relations

$$k_1^+ \wedge k_2^- = h(k)^* k_1^+ \wedge k_2^+ = -h(k) k_1^- \wedge k_2^- \quad (3.14)$$

with the phase

$$h(k) := \frac{k^{+1}}{k^{-2}} = \frac{k^{+2}}{k^{-1}} = 1/h(k)^* \quad (3.15)$$

(identical for all three momenta), the three-point amplitude simplifies to

$$\begin{aligned}\tilde{A}_3^2(k_1, k_2, k_3; \theta) &= -\frac{1}{4}[h(k)^{1/2} e^{i\theta/2} - h(k)^{-1/2} e^{-i\theta/2}]^4 (k_1^+ \wedge k_2^-)^2 \\ &= -\frac{1}{4} e^{2i\theta} [1 - h(k)^{-1} e^{-i\theta/2}]^4 (k_1^+ \wedge k_2^+)^2.\end{aligned}\quad (3.16)$$

We see that the θ dependence factorizes, and the contributions from different instanton sectors differ only by powers of the leg factor $h(k)$. After switching to *real* $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})'$ spinor coordinates, it is easy to see that this three-point tree-level amplitude exactly coincides, in the Leznov frame, with the one obtained from Plebanski's second equation (1.3) for the prepotential ψ . In the Yang frame, one makes contact with Plebanski's first equation. Furthermore, after including the appropriate leg factors the result becomes identical to covariant gauge scattering.

The above structure of the θ dependence is not a speciality of the tree-level three-point function but actually a generic property. One may localize the Maxwell instantons at the worldsheet punctures and thereby define vertex operators V^M , $M = -J, \dots, J$ for various instanton sectors which create an asymptotic string state together with a Maxwell instanton out of the $M=0$ vacuum. Yet, it turns out that any two such operators are proportional to each other, differing merely by (momentum-dependent) normalization,

$$V^M(k) = h(k)^M V(k) \quad (3.17)$$

where $V(k)$ is the vertex operator in the zero-instanton sector. It follows that the partial amplitudes (tree or loop) in the various instanton sectors are related by simple leg factors, and that knowledge of a particular $A^{J,M}$ is sufficient. For this reason, we shall be content to perform our calculations in the zero-instanton sector, except in section four where we employ a Leznov frame.

3.2 Avoiding Contact Terms

The canonical computation of one-loop amplitudes entails the use of the integrated ground state vertex operator in the $(0, 0; 0, 0)$ superconformal ghost picture. Its standard representative is

$$\begin{aligned}\tilde{V}(k) &= \int d^2z d^2\theta d^2\bar{\theta} \exp(ik \cdot X) \\ &= \int d^2z (k^{[+} \cdot \partial x^{-]} - ik^{-} \cdot \psi^{-} k^{-} \cdot \psi^{+}) (k^{[+} \cdot \bar{\partial} x^{-]} + ik^{+} \cdot \bar{\psi}^{-} k^{-} \cdot \bar{\psi}^{+}) e^{ik \cdot x}.\end{aligned}\quad (3.18)$$

The use of this vertex operator in amplitude calculations gives rise to delta functions (and squares of delta functions) on the string worldsheet because of holomorphic/antiholomorphic Wick contractions

$$\langle \partial x^{+a}(z_1) \bar{\partial} \bar{x}^{-b}(z_2) \rangle = \eta^{ab} \delta^{(2)}(z_1 - z_2). \quad (3.19)$$

These contact terms are usually dropped in perturbation theory, but care must be taken to ensure that these terms do not contribute to the scattering in any representation.⁶ It is possible to completely avoid such contact terms by changing the vertex operator representative. Adding the total derivative term

$$-i\partial [(k^{[+}\cdot\bar{\partial}x^{-]} + ik^{+}\cdot\bar{\psi}^{-}k^{-}\cdot\bar{\psi}^{+})e^{ik\cdot x}] - i\bar{\partial} [(k^{[+}\cdot\partial x^{-]} - ik^{-}\cdot\psi^{-}k^{-}\cdot\psi^{+})e^{ik\cdot x}] - \partial\bar{\partial}e^{ik\cdot x} \quad (3.20)$$

and using

$$\partial e^{ik\cdot x} = k^{(+}\cdot\partial x^{-}) e^{ik\cdot x} \quad (3.21)$$

we arrive at

$$V(k) = \int d^2z (2k^{+}\cdot\partial x^{-} - ik^{+}\cdot\psi^{-}k^{-}\cdot\psi^{+}) (2k^{+}\cdot\bar{\partial}x^{-} + ik^{+}\cdot\bar{\psi}^{-}k^{-}\cdot\bar{\psi}^{+}) e^{ik\cdot x} \quad (3.22)$$

which contains x^{+} only in the exponent and therefore precludes not only $\langle\partial x\bar{\partial}x\rangle$ but also $\langle\partial x\partial x\rangle$ and $\langle\bar{\partial}x\bar{\partial}x\rangle$ contractions. In the following we shall use this vertex operator. There is one drawback, however. Since $V(k)$ in (3.22) is no longer invariant under complex conjugation, our computations will not produce holomorphic squares, making chiral splitting impossible.

Next we derive the unintegrated weighted generating functional (Koba-Nielsen form) for n -point amplitudes. The bosonic portion is

$$\prod_{j=1}^n d\theta_j d\bar{\theta}_j \int d\mu_n \exp \left[\int d^2z d^2\tilde{z} J^{+}(z) G(z, \tilde{z}) J^{-}(\tilde{z}) \right]. \quad (3.23)$$

Here, θ_j correspond to an exponentiation

$$k^{+}\cdot\partial x^{-} e^{k\cdot x} = \exp [k\cdot x + \theta k^{+}\cdot\partial x^{-}] \Big|_{\text{multi-linear}} \quad (3.24)$$

of the pre-factor in the vertex operator from which subsequently (after functional integration) the multi-linear part is extracted to obtain the correlation. For the chirally non-split form V in (3.22),⁷ the currents are

$$J^{+}(z) = \sum_{j=1}^n [ik_j^{+}\delta^{(2)}(z-z_j) + \theta_j k_j^{+}\partial\delta^{(2)}(z-z_j) + \bar{\theta}_j k_j^{+}\bar{\partial}\delta^{(2)}(z-z_j)] \quad (3.25)$$

$$J^{-}(z) = \sum_{j=1}^n ik_j^{-}\delta^{(2)}(z-z_j). \quad (3.26)$$

⁶ These contact terms are proportional, after the incorporation of helicity techniques, to inner products $\epsilon_i \cdot \bar{\epsilon}_j$ which vanish manifestly in the MHV amplitudes.

⁷ The real form \tilde{V} of the vertex operator in (3.22) leads to $J^{-}(z)$ being the complex conjugate of (3.25).

The sum in (3.23) may be evaluated to

$$\prod_{j=1}^n d\theta_j d\bar{\theta}_j \int d\mu_n \prod_{i \neq j} \exp \left[-k_i \cdot k_j G_{ij} + i\theta_i k_i^+ \cdot k_j^- \partial G_{ij} + i\bar{\theta}_i k_i^+ \cdot k_j^- \bar{\partial} G_{ij} \right] \quad (3.27)$$

where $G_{ij} = \langle x^+(z_i, \bar{z}_i) x^-(z_j, \bar{z}_j) \rangle$, the bosonic two-point function on the torus, and $d\mu_n$ denotes the measure to integrate over the general punctured super-Riemann surface. The global $N=2$ superspace form generalizing that in (3.27) is

$$\prod_{j=1}^n d\theta_j d\bar{\theta}_j \int d\mu_n^s \prod_{i < j} \exp \left[-k_i \cdot k_j G_{ij} + i\theta_i k_i^+ \cdot k_j^- D_i^+ G_{ij} + i\bar{\theta}_i k_i^+ \cdot k_j^- D_i^- G_{ij} \right] \Big|_{\text{multi-linear}} \quad (3.28)$$

where $d\mu_n^s$ is the superspace measure and D^\pm the $N=2$ superspace derivatives. The form in (3.28) is covariantized in the next section.

3.3 Gauge Invariance and Reference Momenta

In this subsection we describe the transversality of the amplitude at the level of the vertex operators and introduce the calculational tool of reference momenta [39] in order to make manifest the gauge invariance of the amplitudes. These instruments will allow us to compare the integrand with that of IIB superstring and gravity loop amplitudes. Spinor helicity is a useful tool in gauge theory calculations and implicitly has been incorporated in the $N=2$ string, although obscured in previous representations. Here, we find it convenient to switch to a real $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})'$ notation

$$v^{\alpha\dot{\alpha}} = \frac{1}{2} \begin{pmatrix} v^{+1} + v^{-1} - iv^{+2} + iv^{-2} & -iv^{+1} + iv^{-1} + v^{+2} + v^{-2} \\ iv^{+1} - iv^{-1} + v^{+2} + v^{-2} & v^{+1} + v^{-1} + iv^{+2} - iv^{-2} \end{pmatrix} \quad (3.29)$$

for vectors and coordinates and rewrite the $U(1, 1)$ scalar product as ⁸

$$2v^+ \cdot w^- = \epsilon_{\dot{\alpha}\dot{\beta}} \left[v^{+\dot{\alpha}} w^{-\dot{\beta}} - v^{-\dot{\alpha}} w^{+\dot{\beta}} - iv^{+\dot{\alpha}} w^{+\dot{\beta}} - iv^{-\dot{\alpha}} w^{-\dot{\beta}} \right]. \quad (3.30)$$

For a light-like momentum vector $k^{\alpha\dot{\alpha}} = k^\alpha k^{\dot{\alpha}}$, we have the freedom to choose the spinor $q = q(k)$ like ⁹

$$\begin{pmatrix} q_+ \\ q_- \end{pmatrix} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} k_+ \\ k_- \end{pmatrix} \quad \text{hence} \quad q_+ = -iq_- , \quad (3.31)$$

which permits us to express

$$k^+ \cdot v^- = -\frac{1}{2} q_\beta k_\beta v^{\dot{\beta}} \quad (3.32)$$

⁸ Note that the \pm superscripts have different meaning on left- and right-hand sides.

⁹ The matrix is degenerate and not related to the identity by a similarity transformation.

in $SO(2, 2)$ covariant form. Two different spinors q_1^α and q_2^α related to momenta k_1 and k_2 further satisfy

$$q_1^+ = \frac{k_1^+ - ik_1^-}{k_2^+ - ik_2^-} q_2^+ . \quad (3.33)$$

A representation of the two physical polarization vectors $\epsilon_{\alpha\dot{\beta}}^\pm$ in terms of products of spinors is

$$\epsilon_{\alpha\dot{\beta}}^+(k; q) = i \frac{q_\alpha k_{\dot{\beta}}}{q^\gamma k_\gamma} \quad \epsilon_{\alpha\dot{\beta}}^-(k; q) = -i \frac{q_{\dot{\beta}} k_\alpha}{q^\gamma k_{\dot{\gamma}}} \quad (3.34)$$

and has the following properties:

$$\epsilon_{\alpha\dot{\beta}}^\pm(k; \tilde{q}) = \epsilon_{\alpha\dot{\beta}}^\pm(k; q) + f(\tilde{q}, q; k) k_{\alpha\dot{\beta}} , \quad (3.35)$$

$$\epsilon^+ \cdot \epsilon^+ = 0 \quad \epsilon^+ \cdot \epsilon^- = -1 . \quad (3.36)$$

Because the choice of q is arbitrary in any gauge-invariant calculation, it can be chosen to force many inner products to vanish, considerably reducing the amount of algebra in intermediate steps of the calculation.

For example, in an MHV amplitude calculation the individual reference momenta q_i may be taken to coincide: $q_i = q$. This choice eliminates all inner products of polarization vectors,

$$\epsilon^+(k_1; q) \cdot \epsilon^+(k_2; q) = 0 . \quad (3.37)$$

Since individual diagrams must, by dimensional analysis, contain at least one inner product of two polarization vectors, the vanishing of the tree-level MHV (and next to MHV) amplitudes follows immediately. Because the next-to-MHV amplitudes describe the self-dual scattering at tree-level, this also shows the classical triviality of self-dual field theory scattering [9]. At the loop-level it also allows a direct comparison between the N=2 string amplitude calculations and those in the field theory because no $\partial\bar{\partial}G_{ij}$ arises in the integral form in (3.28).

In order to compare we normalize the i^{th} vertex operator with an additional line factor,

$$V'(k_i) = \left(\frac{1}{q_i^\alpha k_{i,\alpha}} \right)^2 V(k_i) \quad (3.38)$$

with q_i satisfying (3.31). By this step, V' takes the same form as the type IIB superstring gravitational vertex operator,

$$V'(k, \epsilon) = \int d^2 z \epsilon_{\alpha\dot{\alpha}}^+ \epsilon_{\beta\dot{\beta}}^+ (\partial x^{\alpha\dot{\alpha}} - i\psi^{\alpha\dot{\alpha}} k^- \cdot \psi^+) (\bar{\partial} x^{\beta\dot{\beta}} + i\bar{\psi}^{\beta\dot{\beta}} k^- \cdot \bar{\psi}^+) e^{ik \cdot x} , \quad (3.39)$$

and is clearly Lorentz covariant due to the reference momenta property in (3.35). The graviton polarization in four dimensions ($d=2+2$) is identified after adjoining $\epsilon_{\alpha\dot{\alpha},\beta\dot{\beta}}^{++}(k) = \epsilon_{\alpha\dot{\alpha}}^+(k)\epsilon_{\beta\dot{\beta}}^+(k)$. Since by (3.17) the vertex operator in a non-zero instanton sector is related to the one in (3.22) by a leg factor only, covariant versions of vertex operators can be given for any instanton sector by an appropriate choice of reference momenta.

The reference momenta defined in (3.31) for the different vertex operators satisfy

$$q_i^\alpha q_{j,\alpha} = 0, \quad (3.40)$$

which means that this choice automatically nullifies all the different inner products $\epsilon^+(k_i; q_i) \cdot \epsilon^+(k_j; q_j) = 0$. Other choices of reference momenta, e.g. $q_j^\alpha = q^\alpha$ for all external lines, may be obtained by a gauge transformation of the vertex operator after normalizing the external lines; they correspond to adding a longitudinal component in (3.35) and yield the same on-shell S-matrix elements.

With the representation in (3.39) the integrand is identical to the Koba-Nielsen representation of the IIB superstring, apart from the spin structure dependence,

$$\int d\mu_n \prod_{i \neq j} \exp(-k_i \cdot k_j G_{ij}) \prod_{i \neq j} \left| \exp \left[\epsilon_{[i} \cdot k_{j]} \partial_i G_{ij} + \epsilon_i \cdot \epsilon_j \partial_i \partial_j G_{ij} + \epsilon_i \cdot \bar{\epsilon}_j \partial_i \bar{\partial}_j G_{ij} \right] \right|_{\text{multi-linear}}^2 \quad (3.41)$$

where the label ‘multi-linear’ means that the integrand is expanded in powers of the polarizations, keeping only the terms linear in each polarization (ϵ_j or $\bar{\epsilon}_j$). The $N=1$ superspace form has

$$\partial_i G_{ij} \rightarrow D_+^i G_{ij} \quad \partial_i \partial_j G_{ij} \rightarrow D_+^i D_+^j G_{ij} \quad \partial_i \bar{\partial}_j G_{ij} \rightarrow D_+^i D_-^j G_{ij}. \quad (3.42)$$

This procedure accounts for the θ integrations in the preceeding form in (3.27), after choosing the reference momenta such that all $\epsilon_i \cdot \epsilon_j = 0$, $\epsilon_i \cdot \bar{\epsilon}_j = 0$ and $\bar{\epsilon}_i \cdot \bar{\epsilon}_j = 0$. The reference momenta that occur naturally in the vertex operator for the $N=2$ string in (3.31) force all inner products in (3.41) $\epsilon_i \cdot \epsilon_j = 0$ and $\epsilon_i \cdot \bar{\epsilon}_j = 0$ via (3.37) and we regain (3.28), although an arbitrary choice of q_i demonstrates the covariance in (3.41).

4 Three-point Genus One

In this section we calculate the genus-one closed-string three-point amplitude originally derived (the $M=0$ part) in [21] and compare the result with field theory, i.e. self-dual gravity. As mentioned in the previous section, in the Leznov frame the tree-level expression $A_3^{J=2}(\theta=0)$ from (3.16) exactly produces the field-theory result generated from the Lagrangians (1.5) or (1.7),

$$A_3^{J=2}(\theta=0) = A_3^{\text{tree}} = (\epsilon_{\dot{\alpha}\dot{\beta}} k_1^{+\dot{\alpha}} k_2^{+\dot{\beta}})^2 \quad (4.1)$$

where we switched to real spinor notation again. Other formulations of self-dual gravity are related by appropriately normalizing the external lines. In the gauge choice of (1.3) and without the external line factors required for covariance, the field-theoretic one-loop expression $A_3^{1\text{-loop}}$ is, by dimensional analysis, constrained to be

$$A_3^{1\text{-loop}} = (k_1^{+\dot{\alpha}} k_{2\dot{\alpha}}^+)^6 \bar{A}_3^{\text{SDG}} . \quad (4.2)$$

This fixes the tensor structure. The remaining proportionality factor \bar{A}_3^{SDG} in the amplitude then boils down to a field-theoretic triangle integral.

The triangle integrals appearing below are infra-red divergent as on-shell kinematics require $k_i^2 = k_i^+ \cdot k_i^- = 0$. The field-theory loop calculation can also be performed by keeping $k_3^2 \neq 0$ until after the integration, which generates the infra-red divergence as $k_3^2 \rightarrow 0$. Direct comparison with the on-shell string scattering is independent of this limit.

After introducing Feynman parameters and Schwinger time, the three-point on-shell one-loop amplitude becomes

$$\begin{aligned} \bar{A}_3^{1\text{-loop}} &= \int \frac{d^d \ell}{(2\pi)^d} \int_0^\infty dT T^2 \int_0^1 da_1 da_2 da_3 \delta(1 - a_1 - a_2 - a_3) \\ &\quad \times \exp \left[-T (a_1 \ell^2 + a_2 (\ell - k_1)^2 + a_3 (\ell + k_3)^2) \right] \\ &\quad \times (\ell^{+\dot{\alpha}} k_{1\dot{\alpha}}^+)^2 \left((\ell - k_1)^{+\dot{\alpha}} k_{2\dot{\alpha}}^+ \right)^2 \left(\ell^{+\dot{\alpha}} k_{3\dot{\alpha}}^+ \right)^2 \end{aligned} \quad (4.3)$$

which, after shifting

$$\ell = \ell' + a_2 k_1 - a_3 k_3 , \quad (4.4)$$

takes the form of (4.2), with

$$\bar{A}_3^{\text{SDG}} = \int \frac{d^d \ell}{(2\pi)^d} \int_0^\infty dT T^2 \int_0^1 da_1 da_2 da_3 a_1^2 a_2^2 a_3^2 \delta(1 - \sum_{j=1}^3 a_j) \exp [-T \ell^2] . \quad (4.5)$$

Integrating over the loop momentum in (unregulated) $d=4$ real dimensions and restoring the tensor structure gives

$$A_3^{1\text{-loop}} = (k_1^{+\dot{\alpha}} k_{2\dot{\alpha}}^+)^6 \times \frac{1}{16\pi^2} \times \frac{1}{15 \times 5!} \int dT . \quad (4.6)$$

The integral is IR divergent,¹⁰ and we regulate it by imposing a Schwinger proper-time cutoff at $T = T_{\text{max}}$; the unregulated results for both the field theory and string theory may be compared without referring to a regulator.

¹⁰ It vanishes in dimensional reduction or regularization.

The three-point function in (4.6) is to be compared with the $N=2$ string result found next. The string-theory calculation in the Leznov frame confirms the same tensor structure as in the field theory,

$$A_3^{J=6}(\theta=0) = (k_1^{+\dot{\alpha}} k_{2\dot{\alpha}}^+)^6 \bar{A}_3^{N=2} \quad (4.7)$$

which differs from the three-point scattering found in [21] only by normalization ($\theta=0$ instead of $M=0$). We may therefore take over their result,

$$\bar{A}_3^{N=2} = \int \frac{d^2\tau}{\tau_2^2} E_3(\tau, \bar{\tau}) , \quad (4.8)$$

where the non-holomorphic Eisenstein series is defined as

$$E_3(\tau, \bar{\tau}) = \sum_{(m,n) \neq (0,0)} \frac{\tau_2^3}{|m + n\tau|^6} \quad (4.9)$$

and satisfies

$$\tau_2^2 \partial_\tau \partial_{\bar{\tau}} E_3(\tau, \bar{\tau}) = 6 E_3(\tau, \bar{\tau}) . \quad (4.10)$$

Using (4.10) the integral in (4.8) can be evaluated,

$$\bar{A}_3^{N=2} = \int d^2\tau (\partial_{\tau_1}^2 + \partial_{\tau_2}^2) E_3(\tau, \bar{\tau}) = \frac{1}{6} \partial_{\tau_2} E_3(\tau, \bar{\tau})|_{\tau_2=\kappa} \quad (4.11)$$

where the integral has been regulated by cutting its large- τ_2 region at $\tau_2 = \kappa$. The over the boundary term at the small- τ_2 end of the fundamental keyhole domain

$$|\tau| \geq 1 \quad \text{and} \quad |\tau_1| \leq \frac{1}{2} \quad (4.12)$$

is zero. For $\tau_2 \rightarrow \infty$, E_3 has the asymptotic form

$$E_3 = 2 \zeta(6) \tau_2^3 + \sqrt{\pi} \zeta(5) \Gamma(5/2) \tau_2^{-2} + O(e^{-2\pi\tau_2}) , \quad (4.13)$$

yielding for (4.11) the expression

$$\bar{A}_3^{N=2} = \zeta(6) \tau_2^2|_{\kappa} \quad (4.14)$$

together with terms that vanish as $\kappa \rightarrow \infty$. Bringing back the tensor structure, we end up with

$$A_3^{J=6}(\theta=0) = (k_1^{+\dot{\alpha}} k_{2\dot{\alpha}}^+)^6 \zeta(6) \tau_2^2|_{\kappa} , \quad (4.15)$$

with $\zeta(6) = \pi^6/945$.

The three-point functions in (4.6) and in (4.15) agree after redefining the string proper time $\tau_2^2 = T$. The regulator

$$T_{\max} = \kappa^2 \quad (4.16)$$

together with a normalization that can be absorbed in (4.16) gives the match.

The two integrals (4.6) and (4.8) differ by a factor of τ_2 or, in the field theory interpretation, a shift in dimension [8]. The matching of the scattering at three-point order is simply a redefinition of the Schwinger proper-time or the cutoff. This is inconsequential at three-point order because the results are both infra-red divergent. However, for the finite higher-point amplitudes such a redefinition is not possible, and the mismatch by a factor of τ_2 makes for a crucial difference between the two theories.

5 Four-Point Genus One

5.1 String Integrand

In this section we analyze the measure for the integration of the four-point (and higher-point) amplitudes for the $N=2$ closed string in the critical dimension $d=2+2$ and compute the integrand in terms of the bosonic and fermionic worldsheet correlators. The field-theory limit is taken in order to compare with the self-dual field theory and one-loop maximally helicity violating amplitudes in gravity. The comparison between the measure factors in the string and field theory persists to multi-genus.

The $N=2$ superconformal algebra has as its generators the energy-momentum tensor T , two supercurrents G^\pm , and the $U(1)$ current J . The associated ghost structure consists of the (b, c) diffeomorphism ghosts, the (β^\mp, γ^\pm) local supersymmetry ghosts, and an additional (b', c') ghost system for the local $U(1)$ invariance or R symmetry. Each chiral $N=2$ matter multiplet $X = (x, \psi)$ and each ghost system contributes a (modular invariant) determinant factor to the one-loop string integration measure (continued to d dimensional target spacetime)

$$Z_d[\alpha]_\beta(\tau, \bar{\tau}) = Z_x(\tau, \bar{\tau}) Z_\psi[\alpha]_\beta(\tau, \bar{\tau}) Z_{bc}(\tau, \bar{\tau}) Z_{\beta\gamma}[\alpha]_\beta(\tau, \bar{\tau}) Z_{b'c'}(\tau, \bar{\tau}) , \quad (5.1)$$

with the respective factors being

$$Z_x(\tau, \bar{\tau}) = \tau_2^{-d/2} |\eta(\tau)|^{-2d} \quad Z_\psi[\alpha]_\beta(\tau, \bar{\tau}) = |\vartheta[\alpha]_\beta(0, \tau)|^d |\eta(\tau)|^{-d} , \quad (5.2)$$

$$Z_{bc}(\tau, \bar{\tau}) = \tau_2 |\eta(\tau)|^4 \quad Z_{\beta\gamma}[\alpha]_\beta(\tau, \bar{\tau}) = |\vartheta[\alpha]_\beta(0, \tau)|^{-4} |\eta(\tau)|^4 \quad (5.3)$$

and, for the one associated with the local $U(1)$ symmetry,

$$Z_{b'c'}(\tau, \bar{\tau}) = \tau_2 |\eta(\tau)|^4 . \quad (5.4)$$

The building blocks are the Jacobi theta functions (featured in the Appendix) with continuous characteristic $[\alpha]_\beta$ equal to spin structure and the Dedekind eta function

$$\eta(\tau) = q^{1/24} \prod_{n \neq 0} (1 - q^n) \quad \text{where} \quad q = e^{2\pi i \tau} , \quad (5.5)$$

with τ denoting the modular parameter of the torus. For general d the product of all determinant factors combines into

$$Z_d[\alpha](\tau, \bar{\tau}) = \tau_2^{-\frac{(d-4)}{2}} |\vartheta[\alpha]_\beta(0, \tau)|^{d-4} |\eta(\tau)|^{-3(d-4)} , \quad (5.6)$$

and equals unity in four real dimensions [20]. This point trivializes the spin structure summation for the one-loop partition function and signals the absence of a tachyonic mode otherwise arising from the q -expansion of eta functions.

Superconformal gauge fixing of the worldsheet $N=(2, 2)$ supergravity produces not only constraints and their ghost systems but also reduces the supergravity path integral to one over the associated finite-dimensional moduli spaces. After explicitly performing the fermionic moduli integrals, which generate picture-raising insertions, one is left with reparametrization and Maxwell moduli. Both come in two varieties: moduli encoding the shape of the $U(1)$ bundle over the worldsheet, and moduli describing the locations and $U(1)$ monodromies of the vertex operators. In the torus case, the former are $(\tau, \bar{\tau})$ and $[\alpha]_\beta$ while the latter comprise $\{(z_i, \bar{z}_i)\}$ and twist angles $\{(\rho_i, \bar{\rho}_i)\}$ interpolating between NS- and R-type puncture.¹¹ Since for genus one the Jacobian torus of spin structures is isomorphic to the worldsheet itself we may parametrize it by an additional torus variable,

$$u = \left(\frac{1}{2} - \alpha\right) \tau + \left(\frac{1}{2} - \beta\right) . \quad (5.7)$$

The modular invariant integration measures are

$$\frac{d^2\tau}{\tau_2^2} \quad \text{and} \quad \frac{d^2u}{\tau_2} \quad (5.8)$$

on the fundamental domain \mathcal{F} of $PSL(2, \mathbb{Z})$ and the torus \mathcal{T} , respectively. Due to spectral flow, the integrand is independent of the twist angles, whose integration thus results merely in a constant volume factor for each puncture. The integration over the puncture locations, however, are nontrivial but modular invariant in the combination $\int_{\mathcal{T}} d^2z V(z, \bar{z})$.

Putting everything together, the scattering amplitude is given by

$$A_n(k_j) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \int_{\mathcal{T}} \frac{d^2u}{\tau_2} \prod_{j=1}^n \int_{\mathcal{T}} d^2z_j \prod_{i < j} e^{-k_i \cdot k_j G_{ij}} K_{KN}(z_i, \bar{z}_i; u, \bar{u}; \tau, \bar{\tau}) , \quad (5.9)$$

with K_{KN} labeling the contractions between the vertex fields.

The expansion of K_{KN} has a suggestive form after the grouping of terms that we now turn to. Each term with fermionic contractions can be paired with a purely bosonic term. This property is a consequence of worldsheet $N=2$ superconformal invariance and can also be used to prove the vanishing of the corresponding tree-level amplitudes.

¹¹ Isometries fix the reparametrization and Maxwell moduli of one of the punctures.

We break the contractions into three groups of terms and analyze the contributions from the string scattering when the reference momenta are chosen to agree with (3.31). These holomorphic and anti-holomorphic mirror terms are depicted graphically in Figures 1 and 2. The bosonic propagator is

$$G_{ij} = -\ln E(z_i - z_j) - \ln E(\bar{z}_i - \bar{z}_j) + \frac{2\pi}{\tau_2} \left[\text{Im}(z_i - z_j) \right]^2 \quad (5.10)$$

where E is the prime form on the torus,

$$E(z, \tau) = \frac{\vartheta_{[1/2]}^{[1/2]}(z, \tau)}{\vartheta_{[1/2]}^{[1/2]}(0, \tau)}, \quad (5.11)$$

and the latter term in (5.10) subtracts the bosonic zero mode from the kernel.¹² The holomorphic half of the fermionic propagator is the Szegő kernel for general continuous monodromies,

$$S_{[\beta]}^{\alpha}(z, \tau) = \frac{\vartheta_{[\beta]}^{\alpha}(z, \tau) \vartheta_{[1/2]}^{[1/2]}(0, \tau)}{\vartheta_{[\beta]}^{\alpha}(0, \tau) \vartheta_{[1/2]}^{[1/2]}(z, \tau)}, \quad (5.12)$$

except for the $\alpha=\beta=1/2$ periodic sector in which an additional zero mode develops. Expansions of the propagators are given in the Appendix.

The first type of term in K_{KN} is

$$\begin{aligned} I^{(1234)} &= k_1^+ \cdot k_2^- k_2^+ \cdot k_3^- k_3^+ \cdot k_4^- k_4^+ \cdot k_1^- \\ &\times \left(\partial_1 G_{12} \partial_2 G_{23} \partial_3 G_{34} \partial_4 G_{41} - S_{12}[\alpha] S_{23}[\alpha] S_{34}[\alpha] S_{41}[\alpha] \right). \end{aligned} \quad (5.13)$$

Its reverse ordering (4321) is denoted by $I^{(4321)}$. The latter gives the complex conjugated contribution via $k^+ \leftrightarrow k^-$. In addition we need the remaining permutations, $I^{(1324)}$, $I^{(1243)}$, and their reverse orderings. This set is closed under permutation of any two indices.

Next we have the three terms

$$\begin{aligned} I^{(12)(34)} &= -k_1^+ \cdot k_2^- k_2^+ \cdot k_1^- k_3^+ \cdot k_4^- k_4^+ \cdot k_3^- \\ &\times \left(\partial_1 G_{12} \partial_2 G_{21} \partial_3 G_{34} \partial_4 G_{43} - S_{12}[\alpha] S_{21}[\alpha] S_{34}[\alpha] S_{43}[\alpha] \right), \end{aligned} \quad (5.14)$$

together with the orderings $I^{(14)(23)}$ and $I^{(24)(13)}$. The terms in (5.14) are products of pairs of Szegő kernels as opposed to the cyclic combinations in (5.13).

¹² Our convention is that G_{ij} marks the full propagator including holomorphic, anti-holomorphic and zero-mode term; the holomorphic piece, $-\ln E(z_i - z_j)$, will be denoted by $G(z_{ij})$, explicitly displaying the holomorphic coordinate.

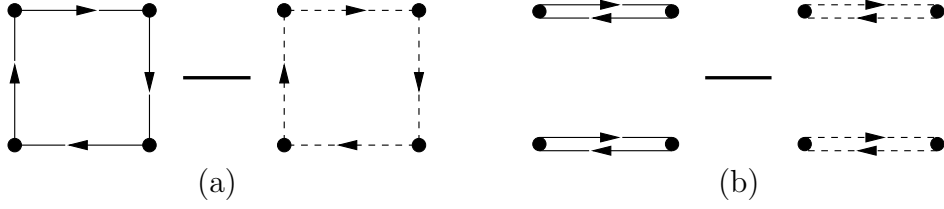


Figure 1: Contributions (a) $I^{(1234)}$ and (b) $I^{(12)(34)}$ to the $N=2$ closed string amplitude. The solid lines are derivatives of the bosonic two-point correlator and the dashed lines are holomorphic (or anti-holomorphic) fermionic Green's functions.

The remaining terms are paired so that there are products of only two Szegő kernels (in a cyclic fashion),

$$\begin{aligned}
I^{(12)} &= k_1^+ \cdot k_2^- k_2^+ \cdot k_1^- \left(\partial_1 G_{12} \partial_2 G_{21} - S_{12}[\alpha_\beta] S_{21}[\alpha_\beta] \right) \\
&\times \left(k_3^+ \cdot k_1^- \partial_3 G_{31} + k_3^+ \cdot k_2^- \partial_3 G_{32} + k_3^+ \cdot k_4^- \partial_3 G_{34} \right) \\
&\times \left(k_4^+ \cdot k_1^- \partial_4 G_{41} + k_4^+ \cdot k_2^- \partial_4 G_{42} + k_4^+ \cdot k_3^- \partial_4 G_{43} \right), \quad (5.15)
\end{aligned}$$

together with its permutations: $I^{(34)}$, $I^{(14)}$, $I^{(23)}$, $I^{(24)}$, and $I^{(13)}$. Terms with three fermion pairs contracted cancel.

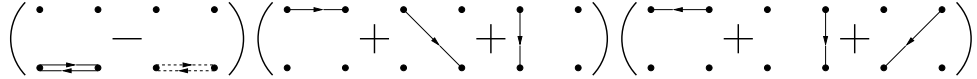


Figure 2: Additional contributions $I^{(34)}$ to the $N=2$ closed string amplitude.

The gauge-invariant vertex operators normalized as in (3.41) produce the same set of terms as in (5.13), (5.14) and (5.15) but with the modification

$$k_i^+ \cdot k_j^- \rightarrow \epsilon_i \cdot k_j \quad (5.16)$$

everywhere. Before and after integrating over spin structures, and with the choice of reference momenta $q_i = q$, this substitution shows that the zero-slope limit of the closed-string amplitude reproduces the Feynman rules of gravity one-loop amplitudes without any $\epsilon_i \cdot \epsilon_j$ or $\epsilon_i \cdot \bar{\epsilon}_j$ terms (i.e. MHV structure).

Let us analyze the structure of K_{KN} in (5.9) given the boson/fermion pairing of the various terms in the expansion. For the periodic spin structure $[\frac{1}{2}]$,

$$S_{ij}[\alpha_\beta] \rightarrow \partial_i G_{ij} \quad (5.17)$$

for each Szegő kernel, and each set of terms in eqs. (5.13), (5.14) and (5.15) vanishes identically. Furthermore, at generic values of $[\alpha_\beta]$ the integrand vanishes at coincident

points $z_i - z_j \rightarrow 0$, making contact with the vanishing tree-level result via worldsheet degeneration. More explicitly, in the short-distance limit of coincident points one gets

$$G(z_i - z_j) = -\ln(z_i - z_j) \quad , \quad S(z_i - z_j) = \frac{1}{z_i - z_j} = -\partial G(z_i - z_j) \quad , \quad (5.18)$$

and the integrand is zero pointwise before integration over the vertex operators. This cancellation can be explained in a number of ways. First, in field theory this is due to the fact that every tree diagram in gauge theory (Yang-Mills or gravity) contains at least one contraction $\epsilon_i \cdot \epsilon_j$, and the identical reference momenta choice for all external lines in an MHV helicity configuration nullifies these terms. Second, at tree-level, target spacetime supersymmetric Ward identities in a supersymmetric gauge theory force the MHV amplitudes to be identically zero (in a supersymmetric gauge theory the tree-level graviton or gauge theory scattering amplitude does not contain internal fermion lines). Third, although the $N=2$ string is not spacetime supersymmetric, the worldsheet $N=2$ superconformal invariance of the vertex operator forces the tree-level amplitude to be zero.

5.2 Comparison with Field Theory at Zero-slope

In this subsection we take the zero-slope limit of the amplitude obtained in the previous subsection and compare it with the field-theory computation obtained in self-dual gravity at one-loop (2.10). Since the integration over the spin structures may be performed before or after the $\alpha \rightarrow 0$ limit and it is not a priori obvious whether the ordering matters (because of singularities at the periodic spin structure), we will examine both orderings: field-theory limit first in the present section, spin structure integration first in the next one. The results will turn out to be the same.

The amplitude from the string differs from self-dual gravity amplitudes in $d=2+2$ because of the (b', c') ghost system associated to the $U(1)$ R symmetry. Thanks to it, the integrand contains an additional τ_2 factor when compared to the integrand of type IIB superstring theory projected onto the self-dual sector of gravity in four dimensions (for example, by toroidal compactification on T^6 to the non-supersymmetric sector). Quite generally, a factor of ¹³

$$\int \frac{d\tau_2}{\tau_2} \tau_2^{n-d/2} e^{-\tau_2 f(k_i)} \quad (5.19)$$

is associated with writing an n -point ϕ^3 Feynman diagram in d dimensions as

$$\int \frac{d^d \ell}{(2\pi)^d} \prod_{j=1}^n \frac{1}{(\ell - p_j)^2} = (4\pi)^{-d/2} \left(\prod_{j=1}^n \int_0^1 da_j \right) \delta(1 - \sum_{j=1}^n a_j) \int_0^\infty dT T^{n-1-d/2} e^{-T f(k_i, a_i)} \quad (5.20)$$

¹³ The n factors of τ_2 , one for every vertex operator, arise from the mapping of the torus to the unit square by $z_i = x_i + \tau y_i$, with $x_i, y_i \in [0, 1]$.

where $f(k_i, a_i) = -(\sum_k p_k a_k)^2 + \sum_k p_k^2 a_k$.

The field-theory limit of the string amplitude is obtained by transforming the string worldsheet coordinates for the vertex operators into a Schwinger proper-time form. From the field-theory point of view, the higher- q terms correspond to the exchange of massive modes (which are absent in the $N=2$ string). We briefly examine the full analytic structure in the limit. Following [40, 41] in the analytic extraction of poles, we introduce new variables w_{ij} satisfying $|w_{ij}| \leq 1$ and defined by

$$w_{ij} = \begin{cases} e^{2\pi i z_{ij}} & \text{for } \text{Im } z_{ij} > 0 \\ q e^{2\pi i z_{ij}} & \text{for } \text{Im } z_{ij} < 0 \end{cases} \quad (5.21)$$

with $z_{ij} \equiv z_i - z_j$. We also make use of the standard parametrization of the vertex insertion points in terms of the real variables α_i and u_i

$$\begin{aligned} u_1 &= y_1 & \alpha_1 &= 2\pi(x_1 + u_1\tau_1) \\ u_2 &= y_2 - y_1 & \alpha_2 &= 2\pi(x_2 - x_1 + u_2\tau_1) \\ u_3 &= y_3 - y_2 & \alpha_3 &= 2\pi(x_3 - x_2 + u_3\tau_1) \\ u_4 &= 1 - y_3 & \alpha_4 &= 2\pi\tau_1 - \alpha_1 - \alpha_2 - \alpha_3 \end{aligned} \quad , \quad (5.22)$$

where $u_1 + u_2 + u_3 + u_4 = 1$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2\pi\tau_1$. This can be achieved by using the translational symmetry of the torus to fix the position of one vertex operator insertion point.

Since only logarithmic derivatives of the prime form multiply the Koba-Nielsen term, multiplying E with a z -independent factor produces only a constant shift in the Koba-Nielsen exponent which vanishes as a result of momentum conservation. We are therefore entitled to neglect constant factors in E and simplify its product representation,

$$E(z_{ij}) = \frac{\vartheta_{[1/2]}^{[1/2]}(z_{ij}, \tau)}{\vartheta_{[1/2]}^{[1/2]}(0, \tau)} \doteq e^{\pi i z_{ij}} \prod_{n=0}^{\infty} (1 - q^n e^{-2\pi i z_{ij}})(1 - q^{n+1} e^{2\pi i z_{ij}}) \quad (5.23)$$

and we define

$$\mathcal{R}(w_{ij}) = \prod_{i \neq j} \prod_{n=0}^{\infty} |1 - w_{ij} q^n|^{-s_{ij}} \quad (5.24)$$

which is the component of $e^{\frac{1}{2}s_{ij}G(z_{ij})}$ that contains all the infinite products from the expansion in (5.23). The remaining contributions from the Koba-Nielsen terms $\prod_{i < j} e^{\frac{1}{2}s_{ij}G(z_{ij})}$ that stem from the $e^{\pi i z_{ij}}$ part of the prime forms and from the zero-mode subtractions in the bosonic Green's functions can be combined into the expression $|q|^{-(su_1 u_3 + tu_2 u_4)}$.

The full amplitude for a given spin structure $[\frac{\alpha}{\beta}]$ may then be rewritten as $A_4[\frac{\alpha}{\beta}](s, t) + A_4[\frac{\alpha}{\beta}](t, u) + A_4[\frac{\alpha}{\beta}](u, s)$, with

$$\begin{aligned}
A_4[\beta]^\alpha(s, t) &= \int_{\mathcal{F}} d^2\tau \, \tau_2^2 \prod_{i=1}^4 \int_0^{2\pi} \frac{d\alpha_i}{2\pi} \delta(2\pi\tau_1 - \Sigma_j \alpha_j) \\
&\times \prod_{i=1}^4 \int_0^1 du_i \delta(1 - \Sigma_j u_j) |q|^{-(su_1u_3+tu_2u_4)} \mathcal{R}(w_{ij}) K_{KN} \quad (5.25)
\end{aligned}$$

and the obvious permutations. We note that, for a given spin structure, K_{KN} is identical to the MHV kinematic factor in IIB superstring theory in (3.41). The function \mathcal{R} , defined from the product expansion of the ϑ -functions as in (5.24), may be expanded in an infinite series as follows:

$$\mathcal{R}(w_{ij}) = \prod_{i=1}^4 |1 - e^{i\alpha_i} |q|^{u_i}|^{-s_i} \sum_{n_i=0}^{\infty} \sum_{|\nu_i| \leq n_i} P_{\{n_i \nu_i\}}^{(4)}(s, t) \prod_{i=1}^4 |q|^{n_i u_i} e^{i\nu_i \alpha_i} . \quad (5.26)$$

Here, $s_i = s$ for i even, $s_i = t$ for i odd, and $P_{\{n_i \nu_i\}}^{(4)}(s, t)$ are polynomials in s and t that may be generated recursively. Consider now the identity

$$\int_0^{2\pi} \frac{d\alpha}{2\pi} e^{i\eta\alpha} |1 - x e^{i\alpha}|^{-s} = x^{-r} \int_0^{\infty} d\beta \, x^\beta \varphi_{r\eta}(s; \beta) , \quad (5.27)$$

where

$$\varphi_{r\eta}(s; \beta) = \sum_{k=0}^{\infty} C_k(s) C_{k+|\eta|}(s) \delta(2k+r+|\eta|-\beta) \quad (5.28)$$

is the inverse Laplace transform of a hypergeometric function, and

$$C_k(s) = \frac{\Gamma(\frac{s}{2} + k)}{\Gamma(\frac{s}{2}) \Gamma(k+1)} . \quad (5.29)$$

For x being some power of $|q|$ we can take advantage of this identity and execute the integration over the angular variables α_i . We observe that the zero-slope limit is equivalent to putting $\mathcal{R}(w_{ij}) = 1$ from the start, since the effect of a nontrivial function \mathcal{R} is felt only at higher order in α' . The analysis can be extended to contain the angular parameters associated with the kinematical factor K_{KN} and justifies the substitution rules that follow from the closed-string context. The remaining propagator terms in K_{KN} generate the Feynman parameters associated with the derivative couplings of the field-theory vertex in the zero-slope limit; the kinematical expression is identical to that obtained in the IIB superstring before summing over spin structures. This procedure has been systematized at one-loop and n -point through the Bern-Kosower string-motivated rules for calculating gauge theory scattering [42] adapted to gravity [43].

If it were not for the non-holomorphic zero-mode part in (5.10), perfect bose-fermi cancellation in eqs. (5.13), (5.14), and (5.15) would occur in the field-theory

limit. As it is, however, the remainder of the various pairings of the bosonic contractions with the fermionic ones is proportional to at least one factor of $(2\pi/i\tau_2)\text{Im } z_{ij}$. The zero modes explicitly break the holomorphicity of the string scattering, and the MHV amplitude may be understood as a holomorphic anomaly in the zero-slope limit of the $N=2$ string.

Remaining in (5.25) is the single factor of τ_2^2 multiplied by bosonic zero-mode contributions from K_{KN} . Comparing with (5.19) we see that this corresponds to a field-theory result in $d=2$ real dimensions [8]. The τ_2 -dependence in the low-energy scattering of gravitons and the $d=2$ technical interpretation follows from either simple toroidal compactifications of the IIB superstring, or using the well-established string-inspired Feynman rules adapted to the case of perturbative gravity [43]. The latter we briefly discuss next in order to map the kinematical structure K_{KN} to the MHV one-loop gravity amplitudes.

The string-inspired generation of the graviton scattering amplitude, which in the zero-slope limit arises from the corners of the moduli space, involves the Feynman-parametrized form originating from a ϕ^3 diagram [43]

$$D = c_n \int_0^1 dx_{i_{n-1}} \dots \int_0^{x_{i_2}} dx_{i_1} \frac{K_{\text{red}}}{\left(\sum_{a < b}^n P_{i_a} \cdot P_{j_b} x_{i_a j_b} (1 - x_{i_a j_b})\right)^{n-d/2}}, \quad (5.30)$$

where, in d dimensions,

$$c_n = (4\pi)^{2-d/2} \frac{\Gamma(n-d/2)}{16\pi^2}. \quad (5.31)$$

P_i is the momentum flowing into the i^{th} leg of the n -gon ϕ^3 diagram, $x_{ij}=x_i-x_j$, and x_i are Feynman parameters. All ϕ^3 diagrams are to be considered with external trees attached where the external lines follow a cycle ordering; in the gravitational case we sum over all of the non-cyclic orderings without associated Yang-Mills color factors. The factor K_{red} comprises terms generated from the kinematical expression identical to the Koba-Nielsen term of the multi-graviton scattering amplitude,

$$\exp \left[(k_i \cdot \epsilon_j - \epsilon_j \cdot k_i) \dot{G}_{ij} - \epsilon_i \cdot \epsilon_j \ddot{G}_{ij} \right] \times (\text{anti} - \text{hol}) \Big|_{\text{multi-linear}}. \quad (5.32)$$

In (5.32) the dotted G s represent worldline derivatives of the complete propagator, including bosonic zero modes.¹⁴ In (5.32) we also have a multiplicative factor of the mixing from holomorphic/anti-holomorphic,

$$\exp [-(\epsilon_i \cdot \bar{\epsilon}_j + \bar{\epsilon}_i \cdot \epsilon_j) H_{ij}], \quad (5.33)$$

where $H_{ij} := \partial_i \bar{\partial}_j G_{ij}$ in the field-theory expression for the amplitude. This comes from the last term in (3.41) and is zero for the $N=2$ string because of the MHV-type condition that $\epsilon_i \cdot \bar{\epsilon}_j = 0$ for all external legs i, j . The propagator in (5.32)

¹⁴ This notation follows that of the first-quantized form of scattering amplitudes at one-loop, derived and motivated by string theory considerations.

is $\dot{G}_{ij} = -\frac{1}{2}\text{sign}(x_{ij}) + x_{ij}$, and the usual Feynman parameters are related to x_i via $x_i = \sum_{j=1}^i a_j$. The point of the form as written in (5.30) is that the kinematical expression arises from the standard first-quantized form of a particle, as generated from integrating over the worldline with measure factor (5.19). Mapping the $N=2$ zero-slope limit to this expression removes the need to explicitly integrate over τ_2 (including four-dimensional box integrals with up to eight insertions of loop momenta in the numerator) because the amplitudes are known [33]. The Feynman-parametrized form of (5.30) and (5.32) produces the integral form of the zero-slope limit of the $N=2$ string expression but with $d=2$ as opposed to $d=4$, as we shall now demonstrate.

We begin by writing all possible ϕ^3 diagrams, obtained by pinching together different sets of vertex operators. Then, after expanding the kinematical factor in (5.32) we collect sets of \dot{G}_{ij} (and \ddot{G}_{ij}) terms in accord with the tree and loop rules (example diagrams are illustrated in Figure 3). The Bern-Kosower tree rule [42] in the low-energy extraction involves substituting on an external leg $-1/(P_i + P_j)^2$ for the occurrence of a single power of \dot{G}_{ij} , from the outside of the diagram into the diagram, and then resubstituting $i = j$ in the remaining momentum flow of the tree-line as well as making the substitution in the remaining \dot{G}_{ij} (and \ddot{G}_{ij}) factors. In the gravity analog we substitute the same when there is a single product of $\dot{G}_{ij}\ddot{G}_{ij}$. In the string-theory amplitude, this amounts to pinching a pole from the $e^{-k_i \cdot k_j G_{ij}}$ kinematical factor (i.e. integrating near $z_i \sim z_j$). After substituting the tree rules on an individual diagram we have a remaining kinematical factor on which we apply the loop rules. The tree rules do not depend on the spacetime spin of the particle being integrated out; rather, the loop rules map to the internal spacetime statistics of the particle.

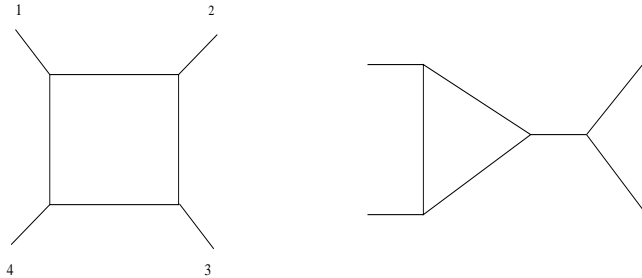


Figure 3: *Example pinched contributions from the string amplitude in the field-theory limit.*

The Bern-Kosower loop rules [42] tell us to expand the holomorphic and anti-holomorphic terms in (5.32) for a given ordering of the four external lines and, after applying the tree rules, to substitute the factors of \dot{G}_{ij} by the low-energy expansion of the propagators with an overall factor of two for combinatorics,

$$\dot{G}_{ij} \rightarrow -\frac{1}{2} \text{sign}(x_{ij}) + x_{ij} . \quad (5.34)$$

These generate the unicyclic contributions and represent the bosonic portion of the worldsheet correlators in the zero-slope limit. Next, we examine the integrand for cyclic occurrences of \dot{G}_{ij} , following the ordering of the legs exiting the loop (for example $\dot{G}_{12}\dot{G}_{23}\dot{G}_{31}$), and substitute as follows,

$$\dot{G}_{ij}\dot{G}_{ji} \rightarrow 2 \quad \text{and} \quad \dot{G}_{i_1i_2}\dot{G}_{i_2i_3}\dots\dot{G}_{i_ni_1} \rightarrow 1 \quad (n > 2) . \quad (5.35)$$

After having applied the cyclic substitutions in (5.35) some \dot{G}_{ij} may be left unsubstituted; they are to be replaced according to (5.34). The outcome is the cyclic contributions and model the zero-slope limit of the fermionic correlations. The loop rules are applied separately on the \dot{G}_{ij} s and \ddot{G}_{ij} s in the case of gravity.

In the case of Einstein-Hilbert gravity, the rules in (5.34) and (5.35) generate a graviton as the state within the loop. In the field-theory limit, the cyclic and unicyclic substitutions arise from the fermionic and bosonic worldsheet propagators, respectively (when there are no \ddot{G}_{ij} terms), and match with the form of K_{KN} . (A worldline systematics in the case of spin $[J \leq 1]$ has also been analyzed in a number of works, including [44] within the context of 1PI diagrams.)

There are further simplifications in the integrand of the MHV amplitudes that are beyond the naive collecting of terms obtained from expanding the Koba-Nielsen factor. The fact that integrating out a spacetime graviton or a spacetime complex scalar in the loop makes no difference for the amplitude implies cancellations of the cyclic terms obtained from the second rule in (5.35). In order to obtain the MHV amplitude with an internal complex scalar, only the first loop rule (5.34) needs to be implemented on both the holomorphic and anti-holomorphic sides.

In the MHV amplitudes, which satisfy (2.8), this means that integrating out the cyclic contributions in (5.35) gives identically zero ($A^{[2]} = A^{[0]}$). This fact has been noted in [43] in the application to a gravitational four-point amplitude with helicity assignment $(--, ++, ++, ++)$. At the level of the superstring and the $N=2$ string this means that the worldsheet fermions do not contribute to the MHV amplitudes in the field-theory limit. Momentum conservation eliminates their total sum; this is demonstrated in the next section. We shall find an identical result when integrating over the spin structures prior to taking $\alpha' \rightarrow 0$.

The q -expansion of the derivative of the bosonic propagator involved in extracting K_{KN} is (see the Appendix)

$$\partial \left[\ln |E(z)|^2 - \frac{2\pi}{\tau_2} (\text{Im } z)^2 \right] = i\pi - \frac{2\pi i}{1 - e^{i\alpha}|q|^u} + \frac{2i\pi}{\tau_2} \text{Im } z + \mathcal{O}(q) , \quad (5.36)$$

where $\text{Im } z \geq 0$. Effectively, after the angular integration [40, 41], the middle term in (5.36) integrates to $-2\pi i$ with higher-order in α' effects (naively there appears to be a potential singularity), yielding the outcome

$$\partial \left[\ln |E(z)|^2 - \frac{2\pi}{\tau_2} (\text{Im } z)^2 \right] \rightarrow -i\pi + \frac{2i\pi}{\tau_2} \text{Im } z = -i\pi(1 - 2y) , \quad (5.37)$$

in agreement with the rules in (5.34) and (5.35) and the tensor algebra of the one-loop diagram after angular integration.¹⁵

Turning to the fermions, one observes that the field-theory limit of a Szegő kernel is independent of the spin structure $[\alpha]_\beta$. In the q -expansion,

$$S[\alpha]_\beta(z_{i_1 i_2}, \tau) S[\alpha]_\beta(z_{i_2 i_3}, \tau) \dots S[\alpha]_\beta(z_{i_n i_1}, \tau) \rightarrow (i\pi)^n + O(q) \quad (5.38)$$

after the integration over the angular coordinates α_i . With this substitution, the second rule in (5.35) obtains for the integral expression of the amplitude. The zero-slope limit of the $N=2$ string reproduces individually *all* the diagrams of the gravity amplitude after a careful tracking of the indices of the \dot{G}_{ij} which come in a specific order within the Koba-Nielsen form in (5.32) ($\dot{G}_{ij} = -\dot{G}_{ji}$).

The primary difference between the integrands of the $N=2$ string and the IIB superstring truncated to obtain four-dimensional gravity lays in the integration measure. Concretely, the $N=2$ string scattering amplitude at n -point has an extra factor of τ_2 compared to the amplitude obtained from a field-theory calculation using the Feynman rules of the self-dual gauge theory [8]. As a result, the amplitude in (2.10) is obtained effectively in $d=2$ and not in $d=4$.

We compare now with the IIB superstring measure, continued to D dimensions. On a torus with spin structure $[\alpha]_\beta$, the NSR fermions and the supersymmetry ghosts (β, γ) produce the determinantal factors

$$Z_\psi[\alpha]_\beta = |\vartheta[\alpha]_\beta(0, \tau)|^D |\eta(\tau)|^{-D} \quad Z_{\beta\gamma}[\alpha]_\beta = |\vartheta[\alpha]_\beta(0, \tau)|^{-2} |\eta(\tau)|^2 \quad (5.39)$$

while the bosonic coordinates and the reparametrization ghosts (b, c) yield

$$Z_x = \tau_2^{-D/2} |\eta(\tau)|^{-2D} \quad Z_{bc} = \tau_2 |\eta(\tau)|^4. \quad (5.40)$$

Together with the Weyl-Peterson measure $d^2\tau/\tau_2^2$, the product $Z_\psi Z_{\beta\gamma} Z_x Z_{bc}$ yields

$$\frac{d^2\tau}{\tau_2^2} \tau_2^{-(D-2)/2} |\vartheta[\alpha]_\beta(0, \tau)|^{D-2} |\eta(\tau)|^{-3(D-2)}, \quad (5.41)$$

not taking into account the factors associated with the vertex operators. Upon compactification on T^{D-d} , it is modified by a lattice sum,

$$Z(\Gamma) = \tau_2^{(D-d)/2} \sum_{(P_L, P_R) \in \Gamma} e^{i\pi\tau P_L \cdot P_L - i\pi\bar{\tau} P_R \cdot P_R} \quad (5.42)$$

where (P_L, P_R) parametrize the (p, q) signature lattice of dimension $D - d$ (consistency requires $P_L^2 - P_R^2 \in 2\mathbb{Z}$ and $p - q \in 8\mathbb{Z}$). Furthermore, the individual vertex operators generate powers of τ_2 after evaluating d^2z (the volume $\int d^2z = \tau_2$).

¹⁵ In the field theory limit $\cot 2\pi z \rightarrow i$.

The two measures, (5.41) times (5.42) for the compactified IIB superstring on one side and (5.6) times (5.8) for the $N=2$ string on the other, differ at zero-slope by a single factor of τ_2 [8]:

$$\frac{d^2\tau}{\tau_2^2} \tau_2^{-(d-2)/2} \longleftrightarrow \frac{d^2\tau}{\tau_2^2} \tau_2^{-(d-4)/2} . \quad (5.43)$$

This calculation indicates the dimensional shift interpretation of the field-theory integration: the IIB superstring compactified on T^6 involves a $d^2\tau/\tau_2^{3-n}$ at n -point (after inserting $D=10$ and $d=4$ in (5.41) and (5.42)). This is the same factor that the bosonic string in $d=26$ compactified on T^{22} generates. In contrast, the $N=2$ string requires a $d^2\tau/\tau_2^{2-n}$.

6 Spin Structure Summation

6.1 Torus Integrals of Elliptic Functions

This section pushes the expression for the full $N=2$ string scattering amplitude a step further and also provides an alternative calculation of its field-theory limit. Concretely, we explicitly evaluate the integrals over the monodromies of the world-sheet fermions, before taking the field theory limit. At four-point order this involves integrating over spin structures various products of up to four holomorphic Szegő kernels (those in (5.13) and (5.14)) together with the anti-holomorphic side with the measure in (5.8).

For a complex structure τ of the torus, we first define (suppressing τ dependence)

$$h_n(\{z_{ij}\}; u) := S^{[\alpha]}_{[\beta]}(z_{12}) S^{[\alpha]}_{[\beta]}(z_{23}) \cdots S^{[\alpha]}_{[\beta]}(z_{n1}) , \quad (6.1)$$

with

$$u = \left(\frac{1}{2} - \alpha\right) \tau + \left(\frac{1}{2} - \beta\right) \quad (6.2)$$

denoting the spin-structure dependent zero locus of the Szegő kernel. By inspecting the zeros and poles of (6.1) we learn how to rewrite this expression in terms of prime forms,

$$h_n = \frac{E(z_{12} - u) E(z_{23} - u) \cdots E(z_{n1} - u)}{[E(-u)]^n E(z_{12}) E(z_{23}) \cdots E(z_{n1})} , \quad (6.3)$$

which exposes the single n th-order pole in u at the origin. The simplest case, $n=2$, yields

$$h_2(z_{12}) = \frac{E(z_{12} - u)^2}{E(u)^2 E(z_{12})^2} = -\wp(z_{12}) + \wp(u) = \partial^2 \ln E(z_{12}) - \partial^2 \ln E(u) \quad (6.4)$$

where $\wp(z)$ is the Weierstraß elliptic function. Furthermore in the coincidence limit $z_{n1} \rightarrow 0$ one observes that $h_n \rightarrow h_{n-1}/z_{n1}$. Since the spin structure has been encoded

in an additional torus variable u , we have to integrate over u (with correct measure) the functions h_n times their anti-holomorphic relatives.

With $h_0 := 1$ we define the integrals

$$f_{n,\bar{n}}(\{z_{ij}, \bar{z}_{ij}\}) := \langle h_n(\{z_{ij}\}; u) \bar{h}_{\bar{n}}(\{\bar{z}_{kl}\}; \bar{u}) \rangle \quad (6.5)$$

with measure

$$\langle \dots \rangle := \int \frac{du \wedge d\bar{u}}{-2i\tau_2} \dots \quad (6.6)$$

which normalizes $\langle 1 \rangle = 1$. Explicit integration for (6.5) is made possible by the following theorem [45]. As $h_n(u)du$ and $\bar{h}_{\bar{n}}(\bar{u})d\bar{u}$ are both closed one-forms with zero residue at $u=0$ we can express the surface integral in terms of period integrals over the a and b cycle,

$$f_{n,\bar{n}} = \frac{i}{2\tau_2} \left[\oint_a h_n(u) du \oint_b \bar{h}_{\bar{n}}(\bar{u}) d\bar{u} - \oint_b h_n(u) du \oint_a \bar{h}_{\bar{n}}(\bar{u}) d\bar{u} \right]. \quad (6.7)$$

We next evaluate the period integrals.

It is a fact [46] that an elliptic function with a single n th-order pole can be expressed as a linear combination of \wp and its derivatives plus a constant. Hence, expanding $h_n(u)$ around the pole (no residue!) we obtain

$$\begin{aligned} h_n(u) &= h_n^{(n)} u^{-n} + \dots + h_n^{(3)} u^{-3} + h_n^{(2)} u^{-2} + h_n^{(0)} + \mathcal{O}(u) \\ &= \frac{(-)^n}{(n-1)!} h_n^{(n)} \wp^{(n-2)}(u) + \dots - \frac{1}{2} h_n^{(3)} \wp'(u) + h_n^{(2)} \wp(u) + H_n^{(0)} \end{aligned} \quad (6.8)$$

with Laurent coefficients $h_n^{(k)}(\{z_{ij}\})$, where we used $\wp(u) = u^{-2} + \mathcal{O}(u^2)$ and

$$\frac{(-)^k}{(k-1)!} \wp^{(k-2)}(u) = u^{-k} + G_k \delta_{k \text{ even}} + \mathcal{O}(u) \quad (6.9)$$

for $k \geq 3$. The holomorphic Eisenstein series

$$G_k = \sum_{(m,n) \neq (0,0)} \frac{1}{(m\tau + n)^k} = 2\zeta(k) + \mathcal{O}(e^{2\pi i\tau}) \quad (6.10)$$

occurring in (6.9) for even k lead to a shift of the constant term in (6.8),

$$h_n^{(0)} \rightarrow H_n^{(0)} = h_n^{(0)} - G_4 h_n^{(4)} - G_6 h_n^{(6)} - \dots - G_{2[\frac{n}{2}]} h_n^{(2[\frac{n}{2}])}. \quad (6.11)$$

The virtue of the expression (6.8) is that the evaluation of its period integrals has become almost trivial. Indeed, since for $k \geq 3$ the antiderivative of $\wp^{(k-2)}(u)$ is $\wp^{(k-3)}(u)$, a doubly-periodic function, the integral of $\wp^{(k-2)}(u)$ over a closed loop vanishes. This observation eliminates all period integrals except for the last two terms in (6.8). Hence, we only require the integrals

$$\oint_a du = 1 \quad \oint_b du = \tau \quad (6.12)$$

as well as

$$\oint_a du \wp(u) = -2\eta_1 = -G_2 \quad \oint_b du \wp(u) = -2\eta_\tau = 2\pi i - G_2\tau \quad (6.13)$$

together with their complex conjugates, where we have introduced the “almost-modular” form (the regulated form of the divergent sum in (6.10) for $k = 2$)

$$G_2(\tau) = 4 - \sum_{(m,n) \neq (0,0)} \frac{1}{(m\tau+n)^2(2m\tau+2n-1)} . \quad (6.14)$$

Via (6.5) and (6.7) this leaves us with only three basic non-vanishing spin structure averages,

$$\langle 1 \rangle = 1 , \quad \langle \wp \rangle = -G_2 + \frac{\pi}{\tau_2} , \quad \langle \wp \bar{\wp} \rangle = G_2 \bar{G}_2 - \frac{\pi}{\tau_2} (G_2 + \bar{G}_2) . \quad (6.15)$$

With these averages we can now compute the integrals (6.5) as

$$\begin{aligned} f_{n,\bar{n}} &= H_n^{(0)} \bar{H}_{\bar{n}}^{(0)} + H_n^{(0)} \bar{h}_{\bar{n}}^{(2)} \langle \bar{\wp} \rangle + \bar{H}_{\bar{n}}^{(0)} h_n^{(2)} \langle \wp \rangle + h_n^{(2)} \bar{h}_{\bar{n}}^{(2)} \langle \wp \bar{\wp} \rangle \\ &= \left[H_n^{(0)} + h_n^{(2)} (-G_2 + \frac{\pi}{\tau_2}) \right] \left[\bar{H}_{\bar{n}}^{(0)} + \bar{h}_{\bar{n}}^{(2)} (-\bar{G}_2 + \frac{\pi}{\tau_2}) \right] - h_n^{(2)} \bar{h}_{\bar{n}}^{(2)} (\frac{\pi}{\tau_2})^2 \end{aligned} \quad (6.16)$$

and observe that they do not split chirally. In the evaluation of the four-point function we require only the cases of $(n, \bar{n}) \in \{(0, 0), (2, 0), (2, 2), (4, 0), (4, 2), (4, 4)\}$ together with the transposes.

It remains to list the coefficients $h_n^{(k)}$. In general $h_n^{(1)} = 0$ and $h_n^{(n)} = (-)^n$. Here, we only need $h_2^{(k)}$ and $h_4^{(k)}$ for even k ,

$$H_2^{(0)} = h_2^{(0)} = -\wp(z_{12}) = \partial^2 \ln E(z_{12}) + G_2 , \quad (6.17)$$

$$h_4^{(2)} = \frac{1}{2} T_4^{-1} \tilde{\partial}^2 T_4 + 2G_2 , \quad (6.18)$$

$$H_4^{(0)} = h_4^{(0)} - G_4 = \frac{1}{24} T_4^{-1} \tilde{\partial}^4 T_4 + G_2 T_4^{-1} \tilde{\partial}^2 T_4 + 2G_2^2 , \quad (6.19)$$

The shorthand notation involving T_4 (generalizable to higher n in this form) is

$$T_4 = E_{12} E_{23} E_{34} E_{41} , \quad (6.20)$$

$$\begin{aligned} \tilde{\partial}^2 T_4 &= E_{12}'' E_{23} E_{34} E_{41} + E_{12} E_{23}'' E_{34} E_{41} + E_{12} E_{23} E_{34}'' E_{41} + E_{12} E_{23} E_{34} E_{41}'' \\ &\quad + 2E_{12}' E_{23}' E_{34} E_{41} + 2E_{12}' E_{23} E_{34}' E_{41} + 2E_{12}' E_{23} E_{34} E_{41}' \\ &\quad + 2E_{12} E_{23}' E_{34}' E_{41} + 2E_{12} E_{23}' E_{34} E_{41}' + 2E_{12} E_{23} E_{34}' E_{41}' \end{aligned} \quad (6.21)$$

and similarly for $\tilde{\partial}^4 T_4$, where we abbreviated $E_{ij} = E(z_{ij})$. The higher $\tilde{\partial}^k$ represents the actions of k derivatives spread out with respect to the insertion points z_{ij} .

For later reference, we present the first few spin structure integrals:

$$f_{2,0} = \partial^2 \ln E_{12} + \frac{\pi}{\tau_2} \quad (6.22)$$

$$f_{2,2} = \left| \partial^2 \ln E_{12} + \frac{\pi}{\tau_2} \right|^2 - \left(\frac{\pi}{\tau_2} \right)^2 \quad (6.23)$$

$$f_{4,0} = \frac{1}{24} T_4^{-1} \tilde{\partial}^4 T_4 + \left(G_2 + \frac{\pi}{\tau_2} \right) \frac{1}{2} T_4^{-1} \tilde{\partial}^2 T_4 + 2 G_2 \frac{\pi}{\tau_2} . \quad (6.24)$$

The analysis above is generalizable to higher genus by employing the prime forms pertaining to the higher-genus Riemann surface.

6.2 Zero-slope Limit

We now analyze the field theory limit of the various terms obtained from summing over the spin structures. In the process of evaluating the ratios of prime forms $E(z_{ij})$ and their derivatives, the following can be implemented:

$$E(z) \rightarrow \frac{1}{2\pi} \sin(2\pi z) , \quad (6.25)$$

which in $f_{n,\bar{n}}$ leads to products of unity and $\cot(2\pi z)$. After the angular integration over z_r , $\cot(2\pi z) \rightarrow i$. Therefore, the analysis of the fermionic correlator terms $f_{n,\bar{n}}$ reduces to combinatoric factors and derivatives of (6.25) with respect to the z -coordinates, together with the appropriate ϕ^3 diagram via pinching the $\prod |E(z_{ij})|^{-\alpha' s_{ij}}$.

We consider first $f_{2,0}(z)$ and $f_{2,2}(z, \bar{z})$ given by (6.22) and (6.23), respectively,

$$f_{2,0}(z) \rightarrow -(2\pi)^2 [1 + \cot^2(2\pi z)] + \frac{\pi}{\tau_2} \rightarrow \frac{\pi}{\tau_2} , \quad (6.26)$$

$$f_{2,2}(z, \bar{z}) \rightarrow \left| -(2\pi)^2 \cot^2(2\pi z) - (2\pi)^2 + \frac{\pi}{\tau_2} \right|^2 - \left(\frac{\pi}{\tau_2} \right)^2 \rightarrow 0 , \quad (6.27)$$

where \bar{z} is a variable independent of z . All of the contributions containing $f_{2,0}$ vanish after adding them up in the expansion of the Koba-Nielsen factor; we will show this after analyzing the remaining terms.

The remaining integrals $f_{n,\bar{n}}$ all involve at least either $n = 4$ or $\bar{n} = 4$. The field theory limit of such a term, exemplified in (6.24), does not vanish individually in the kinematical expression, but like terms add up to zero as we will show below. In the $\tau_2 \rightarrow \infty$ limit the Eisenstein series simplify,

$$G_2 \rightarrow \frac{\pi^2}{3} \quad \text{and} \quad G_4 \rightarrow \frac{\pi^4}{45} . \quad (6.28)$$

As displayed in (6.21) the $\tilde{\partial}^4$ and $\tilde{\partial}^2$ derivatives produce a large number of products of $E_{ij}^{(k)}/E_{ij}$. However, in the field theory limit no z_{ij} dependence survives and

$$\frac{E_{ij}^{(k)}}{E_{ij}} \rightarrow (2\pi i)^k \quad (6.29)$$

which yields

$$T_4^{-1} \tilde{\partial}^4 T_4 \rightarrow +256 (2\pi)^4 \quad T_4^{-1} \tilde{\partial}^2 T_4 \rightarrow -16 (2\pi)^2 . \quad (6.30)$$

Collecting all terms, the net limits of the remaining terms are

$$f_{4,0} \rightarrow \tilde{f}_{4,0} = 10 (2\pi)^4 - \frac{47}{6} (2\pi)^2 \frac{\pi}{\tau_2} \quad (6.31)$$

$$f_{4,2} \rightarrow \tilde{f}_{4,2} = 10 (2\pi)^4 \frac{\pi}{\tau_2} \quad (6.32)$$

$$f_{4,4} \rightarrow \tilde{f}_{4,4} = 100 (2\pi)^8 - \frac{470}{3} (2\pi)^6 \frac{\pi}{\tau_2} . \quad (6.33)$$

It is interesting that a $1/\tau_2$ appears in these terms, which indicates the modification necessary to obtain the MHV amplitudes. These terms do not produce a 0/0 effect as they also vanish in four dimensions, being proportional to the difference between a scalar contribution and a graviton contribution to the MHV amplitude.

The spin structure integrals $f_{n,\bar{n}}$ are being multiplied by kinematical factors $t_{n,\bar{n}}(\{\epsilon_i, k_j\})$ stemming from the contractions of polarization and momentum vectors on four vertex operators (3.39). Since in the field-theory limit all z dependence has dropped from $\tilde{f}_{n,\bar{n}}$, the latter can be factored out from the remaining integrations. This fact allows one to combine directly the various permutations of a given kinematical factor.

We now analyze the kinematical factors $t_{n,\bar{n}} = t_n \bar{t}_{\bar{n}}$. We begin with the contractions of four pairs of fermions, which can happen in two distinct ways. The first possibility is a single cycle connecting all pairs,

$$t_4^{(1234)} = \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_4 \epsilon_4 \cdot k_1 , \quad (6.34)$$

together with permutations $(1 \leftrightarrow 2)$ and $(2 \leftrightarrow 3)$. With arbitrary reference momenta q chosen the same for all polarization vectors, we have

$$t_4^{(1234)} = [12][23][34][41] \quad (6.35)$$

which follows from the substitution

$$\epsilon^{\alpha\dot{\alpha}}(k, q) = i \frac{q^\alpha k^{\dot{\alpha}}}{q^\beta k_\beta} . \quad (6.36)$$

Adding the three permutations produces

$$t_4^{(1234)} + t_4^{(2134)} + t_4^{(1324)} = [12][23][34][41] + [21][13][34][42] + [13][32][24][41] . \quad (6.37)$$

Employing twice the Fierz identity

$$[AB][CD] = [AC][BD] + [AD][CB] \quad (6.38)$$

we find

$$\begin{aligned}
t_4^{(1234)} + t_4^{(2134)} + t_4^{(1324)} &\propto [12][23][34][41] + [13][24]([12][34] + [14][23]) \\
&= [12][23][34][41] + [13]^2[24]^2 \\
&= [12][34]([24][31] + [21][43]) + [13]^2[42]^2 \\
&= -[12][24][43][31] + [12]^2[34]^2 + [13]^2[42]^2 . \quad (6.39)
\end{aligned}$$

Symmetrizing both sides of this equation and using the identity

$$[13]^2[42]^2 + [12]^2[43]^2 + [14]^2[32]^2 = 0 \quad (6.40)$$

(again from (6.38)) one discovers that $t_4^{(1234)} + t_4^{(2134)} + t_4^{(1324)}$ equals minus itself. Thus, the sum in (6.37) vanishes.

The second option for contracting four pairs of fermions produces two cycles of two pairs each,

$$t_4^{(12)(34)} = \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_4 \epsilon_4 \cdot k_3 , \quad (6.41)$$

together with its two permutations. Via (6.36) this equals

$$t_4^{(12)(34)} = [12]^2[34]^2 , \quad (6.42)$$

which upon adding the three permutations and using (6.40) equals zero, too. Similar additions of the field theory limit for the fermionic terms add to zero in the three-point and two-point ϕ^3 diagrams, where the momentum structure involves three and two independent momenta, respectively.

The previous analysis regarding the cyclic terms proportional to $f_{4,0}$ generalizes in a straightforward manner to the remaining cyclic terms multiplying $f_{4,2}$ and $f_{4,4}$. In the field-theory limit all of the z dependence on the holomorphic half of the kinematical expression multiplying these functions is absent (the anti-holomorphic half multiplying $f_{4,2}$ includes bosonic zero modes which translate to Feynman parameters in the field theory limit). After summing over the different contributions on the holomorphic half, these contributions equal zero, as was shown in the preceding paragraphs.

Finally we analyze the $f_{2,0}$ (and $f_{2,2}$) terms. The Wick contractions of two pairs of fermions yield a kinematical factor of

$$t_2^{(ij)} = \epsilon_i \cdot k_j \epsilon_j \cdot k_i , \quad (6.43)$$

which multiplies the remaining kinematical structure from $\partial_k G_{k\ell}$, as displayed by the solid lines in Figure 2. The relation $A^{[0]} = A^{[1]} = A^{[2]}$ enforces all these terms to be zero. This “supersymmetry identity” implies that the fermionic contractions

associated with rule two all generate zero, as discussed in the previous section. In the string amplitude this means that the holomorphic sum of all of the world-sheet fermionic correlators with equal weight add to zero. We have already showed by momentum conservation (for the particular MHV helicity structure) that the four-fermion terms are zero, it follows that the $t_2^{(ij)}$ terms also add to zero (as they all have the same coefficient in (6.26)). This cancellation occurs separately for the four-fermion-pair contractions, i.e. the t^4 terms, as well as the two-fermion-pair contractions, in both two and four dimensions (as the supersymmetry identity holds in both cases). The factors of τ_2 in (6.26) and (6.31)–(6.33) cause a dimensional shift in the integration.

7 Discussion

In this work we have analyzed several aspects of the quantum scattering of the closed $N=(2,2)$ closed superstring at genus one. First, we have derived the zero-slope limit of the one-loop four-point function in the RNS formulation, and explicitly integrated over the spin structures of the worldsheet fermions. We have found agreement with the existing vanishing theorems in the literature. The mapping of the genus-one moduli space integrand to an MHV amplitude at n -point order is performed. Second, we have compared the one-loop integrated three- and higher-point string amplitudes with those of self-dual gravity. The disagreement (vanishing versus nonzero MHV) could be traced to a known [8] difference in the integration measure whose origin is the local R symmetry of the $N=2$ string. Third, we have made manifest the Lorentz and coordinate invariance of the quantum (and classical) scattering by normalizing the vertices and incorporating spinor helicity techniques. Most of this analysis carries over straightforwardly to the open string.

A number of new features have arisen regarding the quantum amplitudes. The $N=2$ string has field equations of self-duality at the classical level, but at genus one its amplitudes are not directly found from self-dual field theory in four dimensions. Rather, the result appears in the loop integration as the dimensionally regulated version of the self-dual amplitudes, continued to two dimensions (with external kinematics in two complex dimensions). These field-theory amplitudes indeed vanish.

The two-dimensional nature of the $N=2$ string loop integration suggests that the effective dynamics of this string is only (real) two-dimensional. Then, the vanishing of two-dimensional gravity (and Yang-Mills) amplitudes may account for the all-order vanishing of the string amplitudes (like at genus one). Clearly, a string in two-dimensional target spacetime has no room for physical excitations (at generic momenta). In four-dimensional spacetime, however, the same situation can be arrived at by increasing the worldvolume dimension from two to four, since a spacetime-

filling brane affords only topological degrees of freedom. Indeed, the analogy

$$T \longleftrightarrow J \qquad (b, c) \longleftrightarrow (b', c') \qquad (7.1)$$

and the fact that the $N=2$ string ghost systems remove two *complex* unphysical directions from the excitation spectrum (best seen for the RNS fermions) have led to the speculation [7] that the $N=2$ string actually is a space-filling brane. It is tempting to interpret the $U(1)^2$ fibre associated with the local R symmetry as carrying the two additional dimensions, making for a total of four parametrizing the full bundle. The $N=2$ string formulation then amounts to a fibration of the 2+2 dimensional world-volume over a Riemann surface.

Another avenue is to search for modifications in the string amplitude which resurrect the non-vanishing four-dimensional MHV scattering. A single factor of $1/\tau_2$ in the integrand of the closed string is required to extract the one-loop self-dual field-theory amplitudes in $d=4$. An insertion of an unintegrated zero-momentum vertex operator or a bosonic zero mode would already do the job, for example, through

$$\lim_{k \rightarrow 0} \sqrt{g} \partial x \bar{\partial} x e^{ik \cdot x} \qquad \text{or} \qquad \partial \bar{\partial} G(z, \bar{z}) = \frac{2\pi}{\tau_2} \delta^{(2)}(z, \bar{z}) . \qquad (7.2)$$

This conformal anomaly may have a target spacetime interpretation as a β function expansion around $d=2$. The insertion of the zero mode, breaking the worldsheet conformal invariance, expands the amplitude to those of one-loop self-dual field theory, and through perturbations of self-duality to gravity and Yang-Mills theory.

We have analyzed one-loop string amplitudes in the field-theory limit, i.e. calculated the leading term in the q -expansion of a string amplitude. As the vanishing theorems and the Ward identities of the $N=2$ string imply that the entire tower of q -expansion coefficients is zero, we expect the above two-dimensional interpretation to hold for the full $N=2$ string theory. A direct verification of the vanishing of the higher- q components of the genus one string amplitude is outside the scope of the present paper; however, we have made important steps in that direction by providing the reader with an explicit expression of the string integrand after spin-structure summation. Whether the analysis remains feasible at $\alpha' \neq 0$ at the level of the full q -expansion is to be shown.

Finally, the one-loop amplitudes generated by the closed $N=2$ string are related through an order $\epsilon=10-d$ identity to those of IIB supergravity in ten dimensions. Via a relation $A_{N=2} \sim \epsilon A_{IIB}$, the zero-slope limit of the $N=2$ string captures the ultraviolet portion of the IIB amplitudes; the latter amplitudes are finite in a dimensionally regulated form in ten dimensions. As both amplitudes are low-energy limits of critical string theories, this suggests a relation between the $N=2$ and $N=1$ strings, which at multi-loop requires a similar relation between the MHV amplitudes and the non-MHV IIB amplitudes. It is interesting to note that membrane-string and string-string connections have been noted in the context of the heterotic $(2, 1)$ formulation in relation to world-volumes of membranes [47, 48].

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8 Appendix : Theta Functions

We list in this Appendix some of the properties of the Jacobi theta functions and elliptic functions useful in this work. The theta function with (α, β) characteristics is defined by the infinite sum

$$\vartheta\left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right](z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i \tau (n+\alpha)^2 + 2\pi i (n+\alpha)(z+\beta)} . \quad (8.1)$$

The theta function satisfies the identity

$$\begin{aligned} \vartheta\left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right](z, \tau) &= e^{\pi i \tau \alpha^2 + 2\pi i \alpha(z+\beta)} \vartheta\left[\begin{smallmatrix}0 \\ 0\end{smallmatrix}\right](z + \tau\alpha + \beta, \tau) \\ &= e^{\pi i \tau (\alpha^2 - 1/2)^2 + 2\pi i (\alpha - 1/2)(z+\beta)} \vartheta\left[\begin{smallmatrix}1/2 \\ 1/2\end{smallmatrix}\right](z + (\alpha - \frac{1}{2})\tau + (\beta - \frac{1}{2}), \tau) \end{aligned} \quad (8.2)$$

with the $\vartheta\left[\begin{smallmatrix}1/2 \\ 1/2\end{smallmatrix}\right](-z, \tau) = -\vartheta\left[\begin{smallmatrix}1/2 \\ 1/2\end{smallmatrix}\right](z, \tau)$. Abbreviating $q = e^{2\pi i \tau}$, the infinite product form of the odd theta function reads

$$\vartheta\left[\begin{smallmatrix}1/2 \\ 1/2\end{smallmatrix}\right](z, \tau) = i q^{1/8} e^{\pi i z} \prod_{n=1}^{\infty} (1 - q^n) \prod_{n=0}^{\infty} [(1 - q^n e^{-2\pi i z}) (1 - q^{n+1} e^{2\pi i z})] , \quad (8.3)$$

and that of its z -derivative at $z=0$ is

$$\vartheta'\left[\begin{smallmatrix}1/2 \\ 1/2\end{smallmatrix}\right](0, \tau) = -2\pi q^{1/8} \prod_{n=1}^{\infty} (1 - q^n)^3 . \quad (8.4)$$

In the same manner, we may rewrite the prime form as

$$E(z, \tau) = \frac{\vartheta\left[\begin{smallmatrix}1/2 \\ 1/2\end{smallmatrix}\right](z, \tau)}{\vartheta'\left[\begin{smallmatrix}1/2 \\ 1/2\end{smallmatrix}\right](0, \tau)} = \frac{e^{\pi i z} \prod_{n=0}^{\infty} [(1 - q^n e^{-2\pi i z}) (1 - q^{n+1} e^{2\pi i z})]}{2\pi i \prod_{n=1}^{\infty} (1 - q^n)^2} , \quad (8.5)$$

where inspection reveals that

$$E(z+1, \tau) = -E(z, \tau) \quad \text{and} \quad E(z+\tau, \tau) = -e^{-\pi i \tau - 2\pi i z} E(z, \tau) . \quad (8.6)$$

The chiral bosonic correlator (without the zero-mode part) is

$$G(z, \tau) = -\ln E(z, \tau) = -\ln \frac{\vartheta_{[1/2]}^{[1/2]}(z, \tau)}{\vartheta_{[1/2]}^{[1/2]}(0, \tau)}. \quad (8.7)$$

Once we insert the bosonic propagators into the expression for the four-point function, the z -independent factors will vanish as a result of momentum conservation. We also need the expanded version of ∂G , where we define the parameter $w = e^{2\pi iz}$,

$$\begin{aligned} \partial G(z, \tau) &= -\pi i \frac{1+w^{-1}}{1-w^{-1}} + 2\pi i \sum_{n=1}^{\infty} q^n \left(\frac{w}{1-q^n w} - \frac{w^{-1}}{1-q^n w^{-1}} \right) \\ &= -\pi \cot(\pi z) + 2\pi i \sum_{n=1}^{\infty} q^n \left(\frac{w}{1-q^n w} - \frac{w^{-1}}{1-q^n w^{-1}} \right). \end{aligned} \quad (8.8)$$

The fermionic Szegő kernel for a given spin structure $(\alpha, \beta) \neq (\frac{1}{2}, \frac{1}{2})$ is

$$\begin{aligned} S_{[\beta]}^{[\alpha]}(z, \tau) &= \frac{\vartheta_{[\beta]}^{[\alpha]}(z, \tau) \vartheta_{[1/2]}^{[1/2]}(0, \tau)}{\vartheta_{[\beta]}^{[\alpha]}(0, \tau) \vartheta_{[1/2]}^{[1/2]}(z, \tau)} \\ &= e^{2\pi i(\alpha-1/2)z} \frac{\vartheta_{[1/2]}^{[1/2]}(z + (\alpha-\frac{1}{2})\tau + (\beta-\frac{1}{2}), \tau) \vartheta_{[1/2]}^{[1/2]}(0, \tau)}{\vartheta_{[1/2]}^{[1/2]}((\alpha-\frac{1}{2})\tau + (\beta-\frac{1}{2}), \tau) \vartheta_{[1/2]}^{[1/2]}(z, \tau)}. \end{aligned} \quad (8.9)$$

For the odd spin structure, the fermionic propagator (again ignoring the zero-mode part) is a derivative of the chiral bosonic Greens function,

$$S_{[1/2]}^{[1/2]}(z, \tau) = -\partial G(z, \tau) = \partial \ln E(z, \tau) = \frac{\vartheta_{[1/2]}^{[1/2]}(z, \tau)}{\vartheta_{[1/2]}^{[1/2]}(z, \tau)}. \quad (8.10)$$

This relation between bosonic and fermionic propagator extends to the anti-holomorphic part and the zero-mode part as well,

$$\frac{2\pi}{i\tau_2} \text{Im } z = \partial_z \frac{2\pi}{\tau_2} [\text{Im } z]^2. \quad (8.11)$$

We must integrate over all spin structures including the odd one; however, the odd spin-structure correlators do not contribute to any of the amplitudes derived in this work.

The Szegő kernels $S_{[\beta]}^{[\alpha]}(z, \tau)$ as well as $\partial G(z, \tau)$ are singular as we take $z \rightarrow 0$; however, the combination

$$\mathcal{F}_{[\beta]}^{[\alpha]}(z, \tau) = S_{[\beta]}^{[\alpha]}(z, \tau) - \partial G(z, \tau) \quad (8.12)$$

is finite in this limit.

The Weierstraß function $\wp(z, \tau)$ is the unique doubly periodic function with a single second-order pole at the origin and no constant term in its Laurent expansion,

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left(\frac{1}{(z-m\tau-n)^2} - \frac{1}{z^2} \right) \\ &= \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) G_{2k+2}(\tau) z^{2k} ,\end{aligned}\tag{8.13}$$

with the modular form

$$G_{2k+2}(\tau) = \sum_{(m,n) \neq (0,0)} (m\tau+n)^{-(2k+2)} = 2\zeta(2k+2) + O(q)\tag{8.14}$$

being known as the holomorphic Eisenstein function of weight $2k+2$, for $k \geq 1$. The antiderivative of the \wp function is denoted a $-\zeta(z)$; it takes the half-point values

$$\zeta(1/2) \equiv \eta_1 = \frac{1}{2}G_2 \quad \text{and} \quad \zeta(\tau/2) \equiv \eta_\tau = \frac{1}{2}G_2\tau - i\pi\tag{8.15}$$

where the failure of the sum in (8.14) to absolutely converge for $k=0$ necessitates a regularized definition of the “almost-modular” form

$$G_2(\tau) = 4 - \sum_{(m,n) \neq (0,0)} \frac{1}{(m\tau+n)^2(2m\tau+2n-1)} = \frac{4\pi}{i} \partial_\tau \ln \eta(\tau) .\tag{8.16}$$

The last equality makes contact with the logarithm of the Dedekind eta function,

$$\ln \eta(\tau) = \sum_{n=1}^{\infty} \ln(1 - e^{2\pi i n \tau}) + i\pi \frac{\tau}{12} ,\tag{8.17}$$

and generalizes to the higher Eisenstein functions, e.g.

$$2G_2(\tau)^2 - 10G_4(\tau) = \left(\frac{4\pi}{i} \right)^2 \partial_\tau^2 \ln \eta(\tau) .\tag{8.18}$$

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